European Journal of Mathematics (2016) 2:304–318
DOI 10.1007/s40879-015-0054-4

RESEARCH ARTICLE

Flexible affine cones over del Pezzo surfaces of degree 4

Jihun Park[1,2] · **Joonyeong Won**[3]

Received: 22 December 2014 / Revised: 6 May 2015 / Accepted: 28 May 2015 /
Published online: 19 June 2015

Abstract For an arbitrary ample divisor A on a smooth del Pezzo surface S of degree 4, we show that the affine cone of S defined by A is flexible.

Keywords Affine cone · Ample divisor · Cylinder · del Pezzo surface · Infinitely transitive action · Flexible variety

Mathematics Subject Classification 14J26 · 14R20 · 14R25

1 Introduction

All considered varieties are assumed to be algebraic and defined over an algebraically closed field of characteristic 0 throughout this article.

The first author has been supported by IBS-R003-G2, Institute for Basic Science in Korea and the second author has been supported by the National Research Foundation in Korea (NRF-2014R1A1A2056432).

✉ Jihun Park
wlog@postech.ac.kr

Joonyeong Won
leonwon@kias.re.kr

1 Center for Geometry and Physics, Institute for Basic Science (IBS), 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk 790-784, Korea

2 Department of Mathematics, POSTECH, 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk 790-784, Korea

3 Algebraic Structure and its Applications Research Center, KAIST, 335 Gwahangno, Yuseong-gu, Daejeon 305-701, Korea

For a positive integer m, a group G is said to *act m-transitively* on a set X if the action is transitive on m-tuples of pairwise distinct points of X. Furthermore, an action is called *infinitely transitive* on X if it is m-transitive for each positive integer m.

For an algebraic variety X, the subgroup of $\mathrm{Aut}(X)$ generated by all algebraic one-parameter unipotent subgroups of $\mathrm{Aut}(X)$ is denoted by $\mathrm{SAut}(X)$. The group $\mathrm{SAut}(X)$ is called the *special automorphism group* of X. Meanwhile, an algebraic variety X is called *flexible* if the tangent space of X at each smooth point $x \in X$ is spanned by the tangent vectors to the orbits $H \cdot x$ of one-parameter unipotent subgroups H of $\mathrm{Aut}(X)$.

The following theorem from [1] shows the connection between flexibility and infinite transitivity.

Theorem 1.1 *Let X be an irreducible affine algebraic variety of dimension at least 2. Then the following are equivalent:*

- *X is flexible;*
- $\mathrm{SAut}(X)$ *acts transitively on the smooth locus of X;*
- $\mathrm{SAut}(X)$ *acts infinitely transitively on the smooth locus of X.*

In addition, [1] proves that every flexible variety is unirational. On the other hand, in [4], it is conjectured that every unirational variety is stably birational to an infinitely transitive variety and it is proved in some cases. Kaliman and Zaidenberg proved that every hypersurface in $\mathbb{A}^{n+2}$ defined by an equation $uv = f(x_1, \ldots, x_n)$ for a non-constant polynomial f has the infinitely transitive property [8]. Moreover, in [3], it is proved that the suspensions over flexible varieties are also flexible. One can find some other examples of flexible varieties in [1,2].

In the present paper we consider affine cones over smooth del Pezzo surfaces polarized by arbitrary ample divisors. For smooth del Pezzo surfaces of degrees less than or equal to 3, the non-existence of $\mathbb{G}_\mathrm{a}$-actions on affine cones by their anticanonical divisors was proved in [5,10]. In [6], the existence and non-existence of $\mathbb{G}_\mathrm{a}$-actions on affine cones over anticanonically polarized del Pezzo surfaces with du Val singularities were fully established according to their singularities and degrees.

Since smooth del Pezzo surfaces of degrees greater than or equal to 6 are toric, affine cones over such surfaces are flexible by [3]. In [11], it is also shown that affine cones over the smooth del Pezzo surface of degree 5 polarized by arbitrary ample divisors are flexible.

Theorem 1.2 *Let S be a smooth del Pezzo surface of degree at least 5. For every ample divisor H, the affine cone*

$$\mathrm{Affcone}_H(S) = \mathrm{Spec} \bigoplus_{m=0}^{\infty} H^0(S, \mathcal{O}_S(mH))$$

is flexible.

In the case of degree 4 the paper [11] proves the flexibility for certain ample divisors including anticanonical divisor. In order to complete the case of degree 4, we prove the following

Main Theorem *Let S be a smooth del Pezzo surface of degree 4. For an arbitrary ample divisor H on S, the affine cone* $\mathrm{Affcone}_H(S)$ *is flexible.*

2 Cylinder, $\mathbb{G}_a$-action and flexibility

Let Y be a projective variety and H be an ample divisor on Y. The following concepts play central role in the study of the flexibility of affine cones over varieties polarized by ample divisors.

Definition 2.1 An open subset U of Y is called a *cylinder* if U is isomorphic to $Z \times \mathbb{A}^1$ for some affine variety Z. A cylinder U is called *H-polar* if the complement of U is the support of an effective $\mathbb{Q}$-divisor that is $\mathbb{Q}$-linearly equivalent to H.

Definition 2.2 A subset W of Y is said to be *invariant with respect to a cylinder* $U = Z \times \mathbb{A}^1$ if $W \cap U = \pi^{-1}(\pi(W))$, where $\pi : U \to Z$ is the projection on the first factor.

Definition 2.3 The variety Y is said to be *transversally covered by cylinders* U_i, $i \in I$, if

- $Y = \bigcup_{i \in I} U_i$;
- there is no proper non-empty subset of Y invariant with respect to all U_i.

Let H be an ample divisor on a smooth projective variety Y. Put

$$\mathrm{Affcone}_H(Y) = \mathrm{Spec} \bigoplus_{m=0}^{\infty} H^0(Y, \mathcal{O}_Y(mH)).$$

The following two theorems show how the concepts above engage in the study of flexibility of the affine cone $\mathrm{Affcone}_H(Y)$.

Theorem 2.4 ([9, Corollary 2.12]) *The affine cone* $\mathrm{Affcone}_H(Y)$ *admits an effective $\mathbb{G}_a$-action if and only if Y contains an H-polar cylinder.*

Theorem 2.5 ([11, Theorem 5]) *If Y has a transversal covering by H-polar cylinders, then the affine cone* $\mathrm{Affcone}_H(Y)$ *is flexible.*

Before we proceed, we present four basic cylinders on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ that will be used in our constructions of transversal coverings.

Example 2.6 Let L_1, L_2, L_3 be three lines on $\mathbb{P}^2$ meeting at a single point. Then $\mathbb{P}^2 \backslash (L_1 \cup L_2 \cup L_3)$ is a cylinder that is isomorphic to an $\mathbb{A}^1$-bundle over a two-point-deleted affine line $\mathbb{A}^1_{**}$.

Example 2.7 Let C be an irreducible conic on $\mathbb{P}^2$ and let L be a line tangent to C. The divisor $C + L$ defines a cylinder isomorphic to an $\mathbb{A}^1$-bundle over a one-point-deleted affine line $\mathbb{A}^1_*$, i.e.,

$$\mathbb{P}^2 \backslash (C \cup L) \cong \mathbb{A}^1_* \times \mathbb{A}^1.$$

Example 2.8 Let C be a cuspidal cubic with a cusp at a point P on $\mathbb{P}^2$ and let T be the Zariski tangent line to C at the point P. Then $\mathbb{P}^2\backslash(C \cup T)$ is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$.

Example 2.9 Let C be an irreducible curve of bidegree (1, 2) on $\mathbb{P}^1 \times \mathbb{P}^1$. There is a curve L of bidegree (1, 0) tangent to the curve C. Let P be the intersection point of C and L and let H be the curve of bidegree (0, 1) that passes through the point P. Then the divisor $C + L + H$ defines a cylinder on $\mathbb{P}^1 \times \mathbb{P}^1$ that is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$. To see this, we take the blow up $\rho\colon S_7 \to \mathbb{P}^1 \times \mathbb{P}^1$ at the point P. Let E be the exceptional curve of ρ and $\widetilde{C}$ be the proper transform of C by ρ. The proper transforms $\widetilde{H}$, $\widetilde{L}$ of H and L by ρ are disjoint (-1)-curves on S_7. By contracting these two (-1)-curves, we obtain a contraction $\psi\colon S_7 \to \mathbb{P}^2$. The curve $\psi(\widetilde{C})$ is an irreducible conic and $\psi(E)$ is a line tangent to $\psi(\widetilde{C})$. Therefore, $C + L + H$ defines a cylinder since

$$\begin{aligned}\mathbb{P}^1 \times \mathbb{P}^1\backslash(C \cup L \cup H) &\cong S_7\backslash\big(\widetilde{C} \cup \widetilde{L} \cup \widetilde{H} \cup E\big)\\ &\cong \mathbb{P}^2\backslash\big(\psi(\widetilde{C}) \cup \psi(E)\big) \cong \mathbb{A}^1_* \times \mathbb{A}^1.\end{aligned}$$

3 Ample divisors on smooth del Pezzo surfaces of degree 4

Let S be a smooth del Pezzo surface of degree 4. It can be obtained by blowing up $\mathbb{P}^2$ at five points in general position. Let $\phi\colon S \to \mathbb{P}^2$ be such a blow up and $E_1, \dots, E_5$ be its exceptional curves. Denote the point $\phi(E_i)$ by P_i.

Let h be the divisor class of S corresponding to $\phi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and e_i be the class of the exceptional curves E_i, where $i = 1, \dots, 5$. Since the classes $h, e_1, \dots, e_5$ form an orthogonal basis of the Picard group of S, for a divisor A on S we may write $[A] = \beta h + \sum_{i=1}^5 \beta_i e_i$, where β and β_i are constants. It is well known that the divisor A is ample if and only if the following inequalities hold

$$\begin{aligned} &\beta \geq -\beta_i > 0 && \text{for} \quad i = 1, 2, 3, 4, 5;\\ &\beta + \beta_i + \beta_j > 0 && \text{for} \quad i \neq j;\\ &2\beta + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 > 0. \end{aligned}$$

In other words, these relations define the ample cone of S. The Mori cone $\overline{\mathbb{NE}}(S)$ of the surface S, the dual of the closure of the ample cone, is polyhedral. Moreover, it is generated by all (-1)-curves on S [7, Theorem 8.2.23].

From now on, the divisor A is always assumed to be ample, unless otherwise stated. The following method to express the divisor A in terms of $-K_S$ and (-1)-curves is adopted from an ongoing joint work of the authors with Cheltsov.

For the log pair (S, A), we define an invariant of (S, A) by

$$\mu = \inf\big\{\lambda \in \mathbb{Q}_{>0} : \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective}\big\}.$$

The invariant μ is always attained by a positive rational number. There is the smallest face $\Delta_{(S,A)}$ of the boundary of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu A$.

Let $\phi\colon S \to Z$ be the contraction given by the face $\Delta_{(S,A)}$. Then either ϕ is a birational morphism or a conic bundle with $Z \cong \mathbb{P}^1$ [7, 8.2.6]. In the former case

$\Delta_{(S,A)}$ is generated by r disjoint (-1)-curves contracted by ϕ, where $r \leq 5$. In the later case $\Delta_{(S,A)}$ is generated by the (-1)-curves in the four reducible fibers of ϕ. Each reducible fiber consists of two (-1)-curves that intersect transversally at one point.

Suppose that ϕ is birational. Let $E_1, \ldots, E_r$ be all (-1)-curves contained in $\Delta_{(S,A)}$. These are disjoint and generate the face $\Delta_{(S,A)}$. Therefore,

$$K_S + \mu A \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i$$

for some positive rational numbers $a_1, \ldots, a_r$ [7, 8.2.6]. We have $a_i < 1$ for every i because $A \cdot E_i > 0$. Vice versa, for every positive rational numbers $a_1, \ldots, a_r < 1$, the divisor

$$-K_S + \sum_{i=1}^{r} a_i E_i.$$

is ample.

Suppose that ϕ is a conic bundle. Then there are a 0-curve B and four disjoint (-1)-curves E_1, E_2, E_3, E_4, each of which is contained in a distinct fiber of ϕ, such that

$$K_S + \mu A \sim_{\mathbb{Q}} aB + \sum_{i=1}^{4} a_i E_i$$

for some positive rational number a and non-negative rational numbers $a_1, a_2, a_3, a_4 < 1$ [7, 8.2.6]. In particular, these curves generate the face $\Delta_{(S,A)}$. Vice versa, for every positive rational number a and non-negative rational numbers $a_1, a_2, a_3, a_4 < 1$ the divisor

$$-K_S + aB + \sum_{i=1}^{r} a_i E_i$$

is ample.

4 Proof of Main Theorem

As before, let S be a smooth del Pezzo surface of degree 4 and A be an ample divisor on S. For the given log pair (S, A), the contraction of the face $\Delta_{(S,A)}$ is either a birational morphism or a conic bundle. We prove that S has transversal coverings by A-polar cylinders in both the cases. We may assume that $\Delta_{(S,A)}$ is a positive dimensional face since we already know that the affine cone over the polarization $(S, -nK_S)$ for every $n \geq 1$ is flexible [11].

4.1 Birational morphism case

We suppose that the contraction $\phi\colon S \to Z$ by the face $\Delta_{(S,A)}$ is birational. There are r disjoint (-1)-curves $E_1, \dots, E_r$ that generate the face $\Delta_{(S,A)}$, where $1 \le r \le 5$. In addition, we can find $5-r$ disjoint (-1)-curves $E_{r+1}, \dots, E_5$ on S that intersect none of the (-1)-curves $E_1, \dots, E_r$. We are then able to obtain a birational morphism $\pi\colon S \to \mathbb{P}^2$ by contracting the five disjoint (-1)-curves $E_1, \dots, E_5$ on S to $\mathbb{P}^2$. Furthermore, we may write

$$K_S + \mu A \sim_{\mathbb{Q}} \sum_{i=1}^{5} a_i E_i,$$

where a_i are rational numbers with $0 < a_i < 1$ for $i = 1, \dots, r$ and $a_i = 0$ for $i = r+1, \dots, 5$.

Denote $\pi(E_i)$ by P_i. Let L_{ij} be the line determined by the points P_i and P_j on $\mathbb{P}^2$ and $\widetilde{L}_{ij}$ be its proper transform by the morphism π. Consider the three intersection points

$$Q_1 = L_{23} \cap L_{45}, \qquad Q_2 = L_{24} \cap L_{35}, \qquad Q_3 = L_{25} \cap L_{34}.$$

We then denote the line determined by the points P_1 and Q_i by L_i and its proper transform on S by $\widetilde{L}_i$. Note that it is possible for two of the lines L_1, L_2, L_3 to coincide, but not for three of them.

Consider the sets

$$\begin{aligned} U_1 &= \pi^{-1}\big(\mathbb{P}^2 \backslash (L_{23} \cup L_{45} \cup L_1)\big),\\ U_2 &= \pi^{-1}\big(\mathbb{P}^2 \backslash (L_{24} \cup L_{35} \cup L_2)\big),\\ U_3 &= \pi^{-1}\big(\mathbb{P}^2 \backslash (L_{25} \cup L_{34} \cup L_3)\big). \end{aligned}$$

These are cylinders isomorphic to $\mathbb{A}^1_{**} \times \mathbb{A}^1$ (Example 2.6).

For a rational number ε we have $-K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1-2\varepsilon)L_1 + (1+\varepsilon)L_{23} + (1+\varepsilon)L_{45}$. Therefore,

$$-K_S \sim_{\mathbb{Q}} (1-2\varepsilon)\widetilde{L}_1 + (1+\varepsilon)\widetilde{L}_{23} + (1+\varepsilon)\widetilde{L}_{45} - 2\varepsilon E_1 + \varepsilon \sum_{i=2}^{5} E_i.$$

Thus we have

$$A \sim_{\mathbb{Q}} \frac{1}{\mu}\Big((1-2\varepsilon)\widetilde{L}_1 + (1+\varepsilon)\widetilde{L}_{23} + (1+\varepsilon)\widetilde{L}_{45} + (a_1 - 2\varepsilon)E_1 + \sum_{i=2}^{5}(a_i+\varepsilon)E_i\Big)$$

By taking $0 < \varepsilon < a_1/2$ we see that U_1 is an A-polar cylinder since

$$U_1 = S \backslash \big(\widetilde{L}_{23} \cup \widetilde{L}_{45} \cup \widetilde{L}_1 \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5\big).$$

Similarly U_2 and U_3 are also A-polar cylinders.

Let C be the conic passing through the points $P_1, \dots, P_5$ and take an arbitrary line L tangent to the conic C. Consider the set

$$U = \pi^{-1}\big(\mathbb{P}^2\backslash(C \cup L)\big),$$

which is a cylinder isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$.

For a rational number ε we have $-K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1-2\varepsilon)L + (1+\varepsilon)C$. Therefore,

$$-K_S \sim_{\mathbb{Q}} (1-2\varepsilon)\widetilde{L} + (1+\varepsilon)\widetilde{C} + \varepsilon\sum_{i=1}^{5} E_i,$$

where $\widetilde{L}$ and $\widetilde{C}$ are the proper transforms of L and C on S. Thus we have

$$A \sim_{\mathbb{Q}} \frac{1}{\mu}\left((1-2\varepsilon)\widetilde{L} + (1+\varepsilon)\widetilde{C} + \sum_{i=1}^{5}(a_i+\varepsilon)E_i\right).$$

By taking $0 < \varepsilon < 1/2$ we see that U is an A-polar cylinder since

$$U = S\backslash\big(\widetilde{L} \cup \widetilde{C} \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5\big).$$

Note that

$$U \cup U_1 \cup U_2 \cup U_3 = S\backslash\big(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5\big).$$

If the lines L_1, L_2, L_3 are distinct, then the three cylinders U_1, U_2, U_3 can cover $S\backslash(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$ without the cylinder U.

Now consider different blow-downs of S that send some E_i onto a line in $\mathbb{P}^2$. Note that for the pair of the disjoint (-1)-curves E_i and E_j, there is a unique (-1)-curve E_{ij} that intersects E_i and E_j but none of the other exceptional curves of π. This is the proper transform of the line joining P_i and P_j by π.

Let $\{i, j, k\}$ be a subset of three elements in $\{1, 2, 3, 4, 5\}$ and let $\{\alpha, \beta\}$ be its complement. Let $\pi_{ijk}\colon S \to \mathbb{P}^2$ be the contraction of the five mutually disjoint (-1)-curves $E_{ij}, E_{jk}, E_{ik}, E_\alpha, E_\beta$. Then the images of the (-1)-curves E_i, E_j, E_k by π_{ijk} are lines on $\mathbb{P}^2$ and

$$\pi_{ijk}(E_i) \cap \pi_{ijk}(E_j) = \pi_{ijk}(E_{ij}), \qquad \pi_{ijk}(E_i) \cap \pi_{ijk}(E_k) = \pi_{ijk}(E_{ik}),$$
$$\pi_{ijk}(E_j) \cap \pi_{ijk}(E_k) = \pi_{ijk}(E_{jk}).$$

There is a unique conic C_{ijk} passing through the points $\pi_{ijk}(E_{ij}), \pi_{ijk}(E_{ik})$, $\pi_{ijk}(E_{jk}), \pi_{ijk}(E_\alpha), \pi_{ijk}(E_\beta)$. Now assume that $a_i \geq a_j \geq a_k$ and take the line $L_{ijk} \subset \mathbb{P}^2$ tangent to the conic C_{ijk} at the point $\pi_{ijk}(E_{ij})$. Put

$$V_{ijk} = \pi_{ijk}^{-1}\big(\mathbb{P}^2\backslash(C_{ijk} \cup L_{ijk})\big).$$

This is a cylinder isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$ (Example 2.7).

For an arbitrary rational number ε

$$-K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1 + a_i + a_k + \varepsilon)C_{ijk} + (1 - a_i - a_k + a_j - 2\varepsilon)L_{ijk} \\ - a_i(\pi_{ijk}(E_i)) - a_j(\pi_{ijk}(E_j)) - a_k(\pi_{ijk}(E_k)),$$

and hence

$$-K_S \sim_{\mathbb{Q}} (1 + a_i + a_k + \varepsilon)\widetilde{C}_{ijk} + (1 - a_i - a_k + a_j - 2\varepsilon)\widetilde{L}_{ijk} \\ + (1 - a_i - \varepsilon)E_{ij} + (a_i - a_j + \varepsilon)E_{jk} + \varepsilon E_{ik} \\ + (a_i + a_k + \varepsilon)E_\alpha + (a_i + a_k + \varepsilon)E_\beta \\ - a_iE_i - a_jE_j - a_kE_k,$$

where $\widetilde{C}_{ijk}$ and $\widetilde{L}_{ijk}$ are the proper transforms of C_{ijk} and L_{ijk} by the morphism π_{ijk}. Thus we have

$$\mu A \sim_{\mathbb{Q}} (1 + a_i + a_k + \varepsilon)\widetilde{C}_{ijk} + (1 - a_i - a_k + a_j - 2\varepsilon)\widetilde{L}_{ijk} \\ + (1 - a_i - \varepsilon)E_{ij} + (a_i - a_j + \varepsilon)E_{jk} + \varepsilon E_{ik} \\ + (a_i + a_k + a_\alpha + \varepsilon)E_\alpha + (a_i + a_k + a_\beta + \varepsilon)E_\beta.$$

Since $a_i < 1$, all coefficients in the divisor above are positive for a sufficiently small positive rational number ε. This shows that V_{ijk} is an A-polar cylinder since

$$S\backslash\big(\widetilde{C}_{ijk} \cup \widetilde{L}_{ijk} \cup E_{ij} \cup E_{jk} \cup E_{ik} \cup E_\alpha \cup E_\beta\big) \cong \mathbb{P}^2\backslash(C_{ijk} \cup L_{ijk}).$$

Since $E_i \subset V_{ijk} \cup V_{i\alpha\beta}$, we have

$$E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \subset \bigcup_{i,j,k=1}^{5} V_{ijk}.$$

Consequently, we have obtained a covering of S by A-polar cylinders:

$$S = U \cup \bigcup_{\ell=1}^{3} U_\ell \cup \bigcup_{i,j,k=1}^{5} V_{ijk}.$$

Now we suppose that there is a non-empty proper subset $W \subset S$ invariant with respect to all cylinders above.

Since the complement $S\backslash W$ is also a non-empty proper subset invariant with respect to all cylinders, up to switching the invariant sets W and $S\backslash W$, we may assume that a point w of W is contained in U_ℓ for some ℓ. Then W must contain the fiber F_w of U_ℓ passing through w. Without loss of generality, we may assume that $\ell = 1$. Then the line $L = \overline{\pi(F_w)} \subset \mathbb{P}^2$ passes through the point Q_1. Since Q_1, Q_2 and Q_3 are not colinear, we may assume that L does not pass through Q_2. Thus all $\mathbb{A}^1$-fibers of the cylinder U_2 intersect L, so that all of them should be contained in W.[1] Therefore, the complement $S\backslash W$ is contained in the support of finitely many curves.

No point of $S\backslash W$ is contained in $U_1 \cup U_2 \cup U_3$: otherwise, the set W, the complement of $S\backslash W$, would be also contained in the support of finitely many curves for the same reason. Therefore, a point w' of $S\backslash W$ is contained either in V_{ijk} for some subset $\{i, j, k\}$ of three elements in $\{1, 2, 3, 4, 5\}$ or in U. Either V_{ijk} or U has a fiber $F_{w'}$ passing through the point w'. If the fiber $F_{w'}$ belongs to V_{ijk}, then $\overline{\pi_{ijk}(F_{w'})}$ is a conic passing through $\pi_{ijk}(w')$ and tangent to L_{ijk} at $\pi_{ijk}(E_{ij})$. If the fiber $F_{w'}$ belongs to U, then $\overline{\pi(F_{w'})}$ is a conic passing through $\pi(w')$ and tangent to L. These show that $S\backslash W$ and $U_1 \cup U_2 \cup U_3$ have a common point. This is a contradiction.

Consequently, the covering of S above is transversal. Therefore, the affine cone $\mathrm{Affcone}_A(S)$ is flexible by Theorem 2.5.

4.2 Conic bundle case

Suppose that the contraction $\phi\colon S \to Z$ given by the face $\Delta_{(S,A)}$ is a conic bundle, i.e., $Z = \mathbb{P}^1$. The face $\Delta_{(S,A)}$ is spanned by an irreducible fiber B of ϕ and four disjoint (-1)-curves E_1, E_2, E_3, E_4. We may then write

$$K_S + \mu A \sim_{\mathbb{Q}} aB + \sum_{i=1}^{4} a_i E_i,$$

where a is a positive rational number and a_i are non-negative rational numbers. Let $\phi_1\colon S \to R$ be the birational morphism obtained by contracting the disjoint (-1)-curves $E_1, \ldots, E_4$.

Subcase 1: R is isomorphic to the Hirzebruch surface $\mathbb{F}_1$.

In this subcase, we have an extra (-1)-curve E_5 which intersects the 0-curve B but none of $E_1, \ldots, E_4$. Contracting $\phi_1(E_5)$, the negative section of $\mathbb{F}_1$, we obtain a birational morphism $\phi_2\colon R \to \mathbb{P}^2$. Put $\pi = \phi_2 \circ \phi_1\colon S \to \mathbb{P}^2$ and $P_i = \pi(E_i)$.

Let C be a cuspidal cubic, with a cusp at P_5, that passes through $P_1, \ldots, P_4$. Let T be the Zariski tangent line to C at P_5. Put

$$U_C = \pi^{-1}\big(\mathbb{P}^2\backslash(C \cup T)\big).$$

It is a cylinder isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$ (Example 2.8).

From $-K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1+\varepsilon)C - 3\varepsilon T$ with an arbitrary rational number ε, we obtain

$$-K_S \sim_{\mathbb{Q}} (1+\varepsilon)\widetilde{C} - 3\varepsilon\widetilde{T} + \varepsilon\sum_{i=1}^{4} E_i + (1-\varepsilon)E_5,$$

[1] This idea originates from [11, Subsection 3.1]

where $\widetilde{C}$ and $\widetilde{T}$ are the proper transforms of C and T, respectively. Since B is linearly equivalent to $\widetilde{T}$,

$$A \sim_{\mathbb{Q}} \frac{1}{\mu}\left((1+\varepsilon)\widetilde{C} + (a-3\varepsilon)\widetilde{T} + \sum_{i=1}^{4}(a_i+\varepsilon)E_i + (1-\varepsilon)E_5\right).$$

By taking a sufficiently small positive rational number ε, we see that the cylinder U_C is A-polar because

$$U_C = S\backslash\left(\widetilde{C} \cup \widetilde{T} \cup E_1 \cup \cdots \cup E_5\right).$$

Let $\mathcal{B}$ be the collection of cuspidal cubics that pass through $P_1, \ldots, P_5$ and whose cusps are located at P_5. Then the cylinders U_C, $C \in \mathcal{B}$, cover S except the exceptional curves $E_1, \ldots, E_5$.

For each (-1)-curve E_i, $i = 1, 2, 3, 4$, we have another (-1)-curve E_i' in the fiber of ϕ that contains E_i. It intersects both E_i and E_5. The curve B is of course linearly equivalent to the divisor $E_i + E_i'$.

Let $\{\alpha, \beta, \gamma\}$ be the complement of the subset $\{i\}$ in $\{1, 2, 3, 4\}$. Let $\pi_i : S \to \mathbb{P}^2$ be the contraction of the five mutually disjoint (-1)-curves $E_{\alpha\beta}$, $E_{\beta\gamma}$, $E_{\alpha\gamma}$, E_i, E_5, where $E_{\alpha\beta}$ is a unique (-1)-curve intersecting E_α, E_β but none of E_γ, E_i, E_5, and $E_{\beta\gamma}$, $E_{\alpha\gamma}$ are defined in the same manner.

The images of the (-1)-curves E_α, E_β, E_γ by π_i are lines on $\mathbb{P}^2$ and

$$\pi_i(E_\alpha) \cap \pi_i(E_\beta) = \pi_i(E_{\alpha\beta}), \qquad \pi_i(E_\beta) \cap \pi_i(E_\gamma) = \pi_i(E_{\beta\gamma}),$$
$$\pi_i(E_\alpha) \cap \pi_i(E_\gamma) = \pi_i(E_{\alpha\gamma}).$$

Moreover $\pi_i(E_i')$ is the conic in $\mathbb{P}^2$ passing through the points $\pi_i(E_{\alpha\beta})$, $\pi_i(E_{\beta\gamma})$, $\pi_i(E_{\alpha\gamma})$, $\pi_i(E_i)$ and $\pi_i(E_5)$. Without loss of generality we may assume that $a_\alpha \geq a_\beta \geq a_\gamma$. We then take the line $L_i \subset \mathbb{P}^2$ tangent to the conic $\pi_i(E_i')$ at the point $\pi_i(E_{\alpha\beta})$. Put

$$V_i = \pi_i^{-1}\left(\mathbb{P}^2\backslash(\pi_i(E_i') \cup L_i)\right).$$

Then V_i is a cylinder on S because $\mathbb{P}^2\backslash(\pi_i(E_i') \cup L_i) \simeq \mathbb{A}^1_* \times \mathbb{A}^1$ (Example 2.7).

As in the birational morphism case, from

$$\begin{aligned} -K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1 + a_\alpha + a_\gamma + \varepsilon)(\pi_i(E_i')) + (1 - a_\alpha - a_\gamma + a_\beta - 2\varepsilon)L_i \\ - a_\alpha(\pi_i(E_\alpha)) - a_\beta(\pi_i(E_\beta)) - a_\gamma(\pi_i(E_\gamma)) \end{aligned}$$

with a sufficiently small positive rational number ε, we obtain

$$\begin{aligned}\mu A \sim_{\mathbb{Q}} (1 + a_\alpha + a_\gamma + a + \varepsilon) E_i' &+ (1 - a_\alpha - a_\gamma + a_\beta - 2\varepsilon) \widetilde{L}_i \\ &+ (1 - a_\alpha - \varepsilon) E_{\alpha\beta} + (a_\alpha - a_\beta + \varepsilon) E_{\beta\gamma} \\ &+ \varepsilon E_{\alpha\gamma} + (a_\alpha + a_\gamma + a + a_i + \varepsilon) E_i \\ &+ (a_\alpha + a_\gamma + \varepsilon) E_5,\end{aligned}$$

where $\widetilde{L}_i$ are the proper transforms of L_i. This shows that V_i is an A-polar cylinder because

$$V_i = S \setminus \big(E_i' \cup \widetilde{L}_i \cup E_{\alpha\beta} \cup E_{\beta\gamma} \cup E_{\alpha\gamma} \cup E_i \cup E_5\big).$$

Moreover, the four cylinders $V_1, \dots, V_4$ cover the (-1)-curves E_1, E_2, E_3 and E_4.

Let $\{i, j\}$ be a subset of two elements in $\{1, 2, 3, 4\}$ and let $\{\alpha, \beta\}$ be its complement. For a pair of the disjoint (-1)-curves E_i and E_j, there is a unique (-1)-curve E_{ij} that intersects E_i and E_j but none of the other exceptional curves of π. As in the birational morphism case, E_{ij} is the proper transform of the line joining P_i and P_j by π.

Let $\pi_{ij} : S \to \mathbb{P}^2$ be the contraction of the five mutually disjoint (-1)-curves $E_{ij}, E_{i5}, E_{j5}, E_\alpha, E_\beta$. Then the images of the (-1)-curves E_i, E_j, E_5 by π_{ij} are lines on $\mathbb{P}^2$ and

$$\pi_{ij}(E_i) \cap \pi_{ij}(E_j) = \pi_{ij}(E_{ij}), \qquad \pi_{ij}(E_i) \cap \pi_{ij}(E_5) = \pi_{i5}(E_{i5}),$$
$$\pi_{ij}(E_j) \cap \pi_{ij}(E_5) = \pi_{ij}(E_{j5}).$$

The image of B by π_{ij} is a line passing through the point $\pi_{ij}(E_{ij})$.

Take the conic C_{ij} passing through the five points $\pi_{ij}(E_{ij}), \pi_{ij}(E_{i5}), \pi_{ij}(E_{j5}), \pi_{ij}(E_\alpha), \pi_{ij}(E_\beta)$. Let L_{ij} be the tangent line to the conic C_{ij} at the point $\pi_{ij}(E_{ij})$. Put

$$V_{ij} = \pi_{ij}^{-1}\big(\mathbb{P}^2 \setminus (C_{ij} \cup L_{ij})\big).$$

It is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$ (Example 2.7).

The union of V_{ij} covers the (-1)-curves $E_1, \dots, E_5$ except the intersection points of E_i and E_{i5}, $i = 1, 2, 3, 4$. In particular, it covers E_5 completely.

To show that V_{ij} is A-polar, we may assume $a_i \geq a_j$ without loss of generality. For an arbitrary positive rational number ε

$$-K_{\mathbb{P}^2} \sim_{\mathbb{Q}} (1 + a_i + \varepsilon) C_{ij} + (1 - a_i + a_j - 2\varepsilon) L_{ij} - a_i(\pi_{ij}(E_i)) - a_j(\pi_{ij}(E_j)).$$

Hence

$$\begin{aligned}-K_S \sim_{\mathbb{Q}} (1 + a_i + \varepsilon) \widetilde{C}_{ij} + (1 - a_i + a_j - 2\varepsilon) \widetilde{L}_{ij} & \\ + (1 - a_i - \varepsilon) E_{ij} + (a_i - a_j + \varepsilon) E_{j5} + \varepsilon E_{i5} & \\ + (a_i + \varepsilon) E_\alpha + (a_i + \varepsilon) E_\beta - a_i E_i - a_j E_j, &\end{aligned}$$

where $\widetilde{C}_{ij}$ and $\widetilde{L}_{ij}$ are the proper transforms of C_{ij} and L_{ij}. Since $\widetilde{L}_{ij}$ is linearly equivalent to B, we have

$$\begin{aligned}\mu A \sim_{\mathbb{Q}} (1+a_i+\varepsilon)\widetilde{C}_{ij} &+ (1-a_i+a_j+a-2\varepsilon)\widetilde{L}_{ij}\\ &+ (1-a_i-\varepsilon)E_{ij} + (a_i-a_j+\varepsilon)E_{j5} + \varepsilon E_{i5}\\ &+ (a_i+a_\alpha+\varepsilon)E_\alpha + (a_i+a_\beta+\varepsilon)E_\beta.\end{aligned}$$

Since $a_i < 1$, all coefficients in the divisor above are positive for a sufficiently small positive rational number ε. This implies that V_{ij} is an A-polar cylinder because

$$V_{ij} = S\backslash\big(\widetilde{C}_{ij} \cup \widetilde{L}_{ij} \cup E_{ij} \cup E_{j5} \cup E_{i5} \cup E_\alpha \cup E_\beta\big).$$

Consequently, we have constructed a covering of S by A-polar cylinders:

$$S = \bigcup_{C\in\mathcal{B}} U_C \cup \bigcup_{i=1}^{4} V_i \cup \bigcup_{i,j=1}^{4} V_{ij}.$$

Now we claim that the covering above is transversal. Suppose that this is not true. Then there is a non-empty proper subset W of S invariant to all cylinders in the covering. Let w be an arbitrary point in W. The image $\pi(W)$ is invariant to the cylinder $\pi(U_C)$ for every $C \in \mathcal{B}$.

Since the complement $S\backslash W$ is also a non-empty proper subset invariant with respect to all cylinders, up to switching the invariant sets W and $S\backslash W$, we may assume that w lies in the cylinder U_C for some $C \in \mathcal{B}$. There is a cuspidal cubic C_w on $\mathbb{P}^2$ that becomes the $\mathbb{A}^1$-fiber of $\pi(U_C)$ passing through the point $\pi(w)$. This is a cuspidal cubic, passing through the point $\pi(w)$ with a cusp at P_5, whose Zariski tangent line at P_5 coincides with that of C. Then the affine curve $C_w\backslash\{P_5\}$ is contained in $\pi(W)$. Choose a cuspidal cubic C' in $\mathcal{B}$ whose Zariski tangent line at P_5 is different from that of C. Then the corresponding cylinder $U_{C'}$ contains all points on C_w except finitely many points. This implies that $S\backslash W$ is contained in the support of finitely many curves.

A point w' in $S\backslash W$ must lie either in V_{ij} for some i, j or in V_i for some i: otherwise W, the complement of $S\backslash W$, would be contained in the support of finitely many curves for the same reason. The cylinder V_{ij} is given by the conic C_{ij} passing through the five points $\pi_{ij}(E_{ij})$, $\pi_{ij}(E_{i5})$, $\pi_{ij}(E_{j5})$, $\pi_{ij}(E_\alpha)$, $\pi_{ij}(E_\beta)$ and the tangent line L_{ij} to the conic C_{ij} at the point $\pi_{ij}(E_{ij})$. The cylinder V_i is defined by the conic $\pi_k(E_i')$ and the line L_i tangent to the conic $\pi_k(E_i')$ at the point $\pi_i(E_{\alpha\beta})$. Therefore, there is either a conic $F_{w'}$ that passes through the point $\pi_{ij}(w')$ and tangent to the line L_{ij} at the point $\pi_{ij}(E_{ij})$ or a conic $G_{w'}$ that passes through the point $\pi_i(w')$ and tangent to the line L_i at the point $\pi_i(E_{\alpha\beta})$. The former (resp. the latter) conic defines the $\mathbb{A}^1$-fiber of V_{ij} (resp. V_i) passing through the point w'. Therefore, $F_{w'}\backslash\{\pi_{ij}(E_{ij})\}$ (resp. $G_{w'}\backslash\{\pi_i(E_{\alpha\beta})\}$) is contained in $\pi_{ij}(S\backslash W)$ (resp. $\pi_i(S\backslash W)$). Since every cuspidal curve that defines an $\mathbb{A}^1$-fiber of U_C intersects $F_{w'}$ and $G_{w'}$ at a point other than $\pi_{ij}(E_{ij})$, $\pi_i(E_{\alpha\beta})$, and P_5, the set W, the complement of $S\backslash W$, is contained in the support of finitely many curves. This is a contradiction.

Subcase 2: R is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$.

The image B by ϕ_1 is a curve of bidegree $(1,0)$ or $(0,1)$ on $\mathbb{P}^1\times\mathbb{P}^1$. We may assume that $\phi_1(B)$ is a curve of bidegree $(0,1)$. There is a one-dimensional family $\mathcal{C}$ of irreducible curves of bidegree $(1,2)$ passing through the four points $\phi_1(E_1),\dots,\phi_1(E_4)$. For each curve C in the family $\mathcal{C}$, we take a curve L_C of bidegree $(1,0)$ tangent to C at a point. By H_C denote the curve of bidegree $(0,1)$ that passes through the intersection point of C and L_C. These three curves C, L_C and H_C define a cylinder isomorphic to an $\mathbb{A}^1$-bundle over $\mathbb{A}^1_*$ (Example 2.9). Put

$$U_C=\phi_1^{-1}\big(\mathbb{P}^1\times\mathbb{P}^1\backslash(C\cup L_C\cup H_C)\big).$$

For an arbitrary rational number ε

$$-K_{\mathbb{P}^1\times\mathbb{P}^1}\sim_{\mathbb{Q}}(1+\varepsilon)C+(1-\varepsilon)L_C+(a-2\varepsilon)H_C-a\phi_1(B).$$

Now let δ_i be 1 if L_C passes through $\phi(E_i)$ and 0 if not. Then we obtain

$$\begin{aligned}-K_S\sim_{\mathbb{Q}}(1+\varepsilon)\widetilde{C}+(1-\varepsilon)\widetilde{L}_C+(a-2\varepsilon)\widetilde{H}_C-aB\\ +\sum_{i=1}^{4}\big(\varepsilon+(1+a-3\varepsilon)\delta_i\big)E_i,\end{aligned}$$

where $\widetilde{C}$, $\widetilde{L}_C$ and $\widetilde{H}_C$ are the proper transforms of C, L_C and H_C, respectively. Thus we have

$$\begin{aligned}A\sim_{\mathbb{Q}}\frac{1}{\mu}\Bigg((1+\varepsilon)\widetilde{C}+(1-\varepsilon)\widetilde{L}_C+(a-2\varepsilon)\widetilde{H}_C\\ +\sum_{i=1}^{4}\big(a_i+\varepsilon+(1+a-3\varepsilon)\delta_i\big)E_i\Bigg).\end{aligned}$$

By taking a sufficiently small positive rational number ε, we can see that U_C is an A-polar cylinder since

$$U_C=S\backslash\big(\widetilde{C}\cup\widetilde{L}_C\cup\widetilde{H}_C\cup E_1\cup E_2\cup E_3\cup E_4\big).$$

The cylinders defined by C in $\mathcal{C}$ in this manner cover S except E_1, E_2, E_3, E_4.

For each (-1)-curve E_i, we have another (-1)-curve E_i' in the fiber of ϕ that contains E_i. In addition, there is a (-1)-curve E_i'' that intersects E_i but none of the other exceptional curves of ϕ_1. The curve B is linearly equivalent to the divisor E_i+E_i'.

Let $\{\alpha,\beta,\gamma\}$ be the complement of the subset $\{i\}$ in $\{1,2,3,4\}$. The (-1)-curves E_α, E_β, E_γ, E_i', E_i'' are mutually disjoint. Let $\pi_i: S\to\mathbb{P}^2$ be the contraction of the five mutually disjoint (-1)-curves $E_{\alpha\beta}$, $E_{\beta\gamma}$, $E_{\alpha\gamma}$, E_i', E_i'', where $E_{\alpha\beta}$ is a unique (-1)-curve intersecting E_α, E_β but none of E_γ, E_i', E_i'', and $E_{\beta\gamma}$, $E_{\alpha\gamma}$ are defined in the same manner.

The images of the (-1)-curves E_α, E_β, E_γ by π_i are lines on $\mathbb{P}^2$ and

$$\pi_i(E_\alpha)\cap\pi_i(E_\beta)=\pi_i(E_{\alpha\beta}),\qquad \pi_i(E_\beta)\cap\pi_i(E_\gamma)=\pi_i(E_{\beta\gamma}),$$
$$\pi_i(E_\alpha)\cap\pi_i(E_\gamma)=\pi_i(E_{\alpha\gamma}).$$

Moreover $\pi_i(E_i)$ is the conic in $\mathbb{P}^2$ passing through the points $\pi_i(E_{\alpha\beta}), \pi_i(E_{\beta\gamma})$, $\pi_i(E_{\alpha\gamma}), \pi_i(E_i')$ and $\pi_i(E_i'')$. Without loss of generality we may assume that $a_\alpha \geq a_\beta \geq a_\gamma$. We then take the line $L_i \subset \mathbb{P}^2$ tangent to the conic $\pi_i(E_i)$ at the point $\pi_i(E_{\alpha\beta})$. Put

$$V_i=\pi_i^{-1}\big(\mathbb{P}^2\backslash(\pi_i(E_i)\cup L_i)\big).$$

Then V_i is a cylinder on S because $\mathbb{P}^2\backslash(\pi_i(E_i)\cup L_i)\simeq\mathbb{A}^1_*\times\mathbb{A}^1$ (Example 2.7).

As before, for a sufficiently small positive rational number ε,

$$\begin{aligned}-K_{\mathbb{P}^2}\sim_{\mathbb{Q}}(1+a_\alpha+a_\gamma+\varepsilon)(\pi_i(E_i))+(1-a_\alpha-a_\gamma+a_\beta-2\varepsilon)L_i\\ -a_\alpha(\pi_i(E_\alpha))-a_\beta(\pi_i(E_\beta))-a_\gamma(\pi_i(E_\gamma))\end{aligned}$$

yields an effective $\mathbb{Q}$-divisor

$$\begin{aligned}\mu A\sim_{\mathbb{Q}}(1+a_\alpha+a_\gamma+a+a_i+\varepsilon)E_i+(1-a_\alpha-a_\gamma+a_\beta-2\varepsilon)\widetilde{L}_i\\ +(1-a_\alpha-\varepsilon)E_{\alpha\beta}+(a_\alpha-a_\beta+\varepsilon)E_{\beta\gamma}\\ +\varepsilon E_{\alpha\gamma}+(a_\alpha+a_\gamma+a+\varepsilon)E_i'\\ +(a_\alpha+a_\gamma+\varepsilon)E_i'',\end{aligned}$$

where $\widetilde{L}_i$ are the proper transforms of L_i. This shows that V_i is an A-polar cylinder because

$$V_i=S\backslash\big(E_i\cup\widetilde{L}_i\cup E_{\alpha\beta}\cup E_{\beta\gamma}\cup E_{\alpha\gamma}\cup E_i'\cup E_i''\big).$$

Moreover, the four cylinders V_i cover the (-1)-curves E_1, E_2, E_3 and E_4.

We have obtained a covering of S by A-polar cylinders:

$$S=\bigcup_{C\in\mathcal{C}}U_C\cup\bigcup_{i=1}^{4}V_i.$$

Finally we claim that the covering above is transversal. Suppose that it is not transversal. Then there is a non-empty proper subset W of S that is invariant with respect to all cylinders in the covering above. Let w be a point in W.

As in the previous cases, we may assume that w belongs to U_C for some $C\in\mathcal{C}$. The set $\phi_1(W)\subset\mathbb{P}^1\times\mathbb{P}^1$ is invariant to the cylinder $\phi_1(U_C)$. Let P be the intersection point of C and L_C. Then there is a curve C_w of bidegree $(1,2)$ tangent to L_C at the point P and passing through the point $\phi_1(w)$. Then the affine curve $C_w\backslash\{P\}$ defines the $\mathbb{A}^1$-fiber in the cylinder U_C passing through the point w. Therefore, $\phi_1^{-1}(C_w \backslash \{P\})$ is contained in W. We can choose a curve C' in $\mathcal{C}$ that intersects C_w at four distinct points. Therefore, every $\mathbb{A}^1$-fiber in $U_{C'}$ except finitely many fibers must be contained in W. This shows that its complement $S \backslash W$ is contained in the support of finitely many curves.

A point in $S \backslash W$ cannot belong to the cylinder U_C for any $C \in \mathcal{C}$: otherwise W would be contained in the support of finitely many curves. Therefore, a point w' in $S \backslash W$ is contained in V_i for some i. Then there is a conic $C_{w'}$ on $\mathbb{P}^2$ tangent to L_i at the point $\pi_i(E_{\alpha\beta})$ and passing through the point $\pi_i(w')$. This conic defines the $\mathbb{A}^1$-fiber of V_i passing through the point w'. Therefore, $C_{w'} \backslash \{\pi_i(E_{\alpha\beta})\}$ is contained in $\pi_i(S \backslash W)$. Since every curve of bidegree $(1, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ that defines an $\mathbb{A}^1$-fiber of U_C intersects the curve $\phi_1(\pi_i^{-1}(C_{w'}))$ of bidegree $(3, 3)$ outside $\phi_1(E_{\alpha\beta})$, the set W, the complement of $S \backslash W$, is contained in the support of finitely many curves. This is a contradiction.

This completes the proof of Main Theorem.

References

1. Arzhantsev, I., Flenner, H., Kaliman, S., Kutzschebauch, F., Zaidenberg, M.: Flexible varieties and automorphism groups. Duke Math. J. **162**(4), 767–823 (2013)
2. Arzhantsev, I., Flenner, H., Kaliman, S., Kutzschebauch, F., Zaidenberg, M.: Infinite transitivity on affine varieties. In: Bogomolov, F., Hassett, B., Tschinkel, Yu. (eds.) Birational Geometry, Rational Curves, and Arithmetic, pp. 1–13. Springer, New York (2013)
3. Arzhantsev, I.V., Kuyumzhiyan, K.G., Zaidenberg, M.G.: Flag varieties, toric varieties, and suspensions: three instances of infinite transitivity. Sb. Math. **203**(7), 923–949 (2012)
4. Bogomolov, F., Karzhemanov, I., Kuyumzhiyan, K.: Unirationality and existence of infinitely transitive models. In: Bogomolov, F., Hassett, B., Tschinkel, Yu. (eds.) Birational Geometry, Rational Curves, and Arithmetic, pp. 77–92. Springer, New York (2013)
5. Cheltsov, I., Park, J., Won, J.: Affine cones over smooth cubic surfaces (2013). J. Eur. Math. Soc. (to appear)
6. Cheltsov, I., Park, J., Won, J.: Cylinders in singular del Pezzo surfaces (2013). arXiv:1311.5257
7. Dolgachev, I.V.: Classical Algebraic Geometry. Cambridge University Press, Cambridge (2012)
8. Kaliman, S., Zaidenberg, M.: Affine modification and affine hypersurfaces with a very transitive automorphism group. Transform. Groups **4**(1), 53–95 (1999)
9. Kishimoto, T., Prokhorov, Yu., Zaidenberg, M.: $\mathbb{G}_a$-actions on affine cones. Transform. Groups **18**(4), 1137–1153 (2013)
10. Kishimoto, T., Prokhorov, Yu., Zaidenberg, M.: Unipotent group actions on del Pezzo cones. Algebraic Geom. **1**(1), 46–56 (2014)
11. Perepechko, A.Yu.: Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5. Funct. Anal. Appl. **47**(4), 284–289 (2013)