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Visualizing Quaternion Multiplication

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ABSTRACT Quaternion rotation is a powerful tool for rotating vectors in 3-D; as a result, it has been used in various engineering fields, such as navigations, robotics, and computer graphics. However, understanding it geometrically remains challenging, because it requires visualizing 4-D spaces, which makes exploiting its physical meaning intractable. In this paper, we provide a new geometric interpretation of quaternion multiplication using a movable 3-D space model, which is useful for describing quaternion algebra in a visual way. By interpreting the axis for the scalar part of quaternion as a 1-D translation axis of 3-D vector space, we visualize quaternion multiplication and describe it as a combined effect of translation, scaling, and rotation of a 3-D vector space. We then present how quaternion rotation formulas and the derivative of quaternions can be formulated and described under the proposed approach.

INDEX TERMS 4-dimensional spaces, geometry, scaling, quaternion rotation.

I. INTRODUCTION

A quaternion is a 4-tuple number system invented by Hamilton in 1844. As an extension of the complex number system, quaternions are comprised of a scalar and three imaginary numbers, and are generally represented in the form of their linear combinations [1]. Although they were invented to describe 3-dimensional transformations, they have not been widely used in engineering owing to difficulties in exploiting their physical meaning. Instead, with the development of linear algebra, more-intuitive methods such as matrix multiplication and Euler angle were widely used. However, the latter methods have some problems such as gimbal lock and large computational requirements. Compared with these methods, quaternion rotation requires fewer computations and represents sequential rotations more simply. Because of these advantages, quaternion rotation has replaced the other methods in many fields such as computer graphics [2], computer simulations [3], robotics, attitude control [4], and orbital mechanics [5].

Unfortunately, however, visually interpreting quaternion rotation is known to be challenging because it requires the portrayal of 4-dimensions. As a result, even today many researchers or engineers resort to the other methods even though they know utilizing quaternions would be more efficient. Although some models or explanation methods (e.g., mass-points model, a pair of mutually orthogonal planes model [6], twisting belt [7], rolling ball [8]) have been suggested to aid in intuitively explaining quaternion rotation, they are still based on somewhat abstract and static images of four-dimensions and are unable to represent the entire dynamic process during transformation regarding quaternion multiplication, which means that exploiting the physical meaning remains difficult.

This paper introduces a new geometric interpretation of quaternions and a movable 3-dimensional space model, and then presents geometric derivations of rotational formulae and the derivative of quaternions. The proposed model represents four-dimensions by assigning the scalar part to a translational axis of the 3-dimensional vector space; thereby, the method can be said to be more practical for representing physical transformations regarding quaternion multiplication compared to prior models [6]–[8]. In this study, our model represents quaternion multiplication as translations, reductions (or expansions), and rotations for 3-dimensional vector spaces, and then derives magnitude-invariant rotation formulae by compensating for scaling of axes.

Because the proposed geometric approach minimizes the requirement of constructing abstract space partitions from rigorous mathematical logics, it is a clearer way to visualize quaternion multiplication than the existing ways. Further, the geometric derivations of rotation formulae, based on this approach, help researchers intuitively discover why there is a need for a sandwiching multiplication form rather than a simple multiplication form; moreover, it would encourage them to find more applications of quaternion algebra in their research fields.

The remainder of this paper is organized as follows. Section II provides a brief introduction to quaternion algebra. Section III presents a geometric model of quaternions, and Section IV presents visualization of quaternion multiplication along with its interpretation. Section V derives rotation formulae. Section VI derives the derivative of quaternions. Finally, Section VII concludes the paper.

II. PRELIMINARIES: FUNDAMENTAL ALGEBRA OF QUATERNION

This paper is focused on the visualization of quaternion multiplication. Thus, among algebraic properties of quaternion, multiplication is the main theme in this paper. However, first, we present some algebraic definitions or properties of quaternions (addition, complex conjugate, norm, and inverse) to aid in understanding quaternion multiplication. Accordingly, in this section, we briefly introduce the fundamental algebra of quaternions.

As an extension of complex numbers, a quaternion $q(q_0, \boldsymbol{q})$ is comprised of a scalar q_0 and a vector $\boldsymbol{q} = (q_1, q_2, q_3)$, and represented as

$$q(q_0, \boldsymbol{q}) = (q_0, q_1, q_2, q_3) = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$

with three imaginary numbers, $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$. It should be noted that $\boldsymbol{q}$ also implies $\boldsymbol{q} = q_1\boldsymbol{i}+q_2\boldsymbol{j}+q_3\boldsymbol{k}$, to avoid any confusion. If the scalar part is zero, then the quaternion is usually called a pure quaternion.

The addition of two quaternions is performed componentwise; e.g., addition of two quaternions $q = q(q_0, \boldsymbol{q})$ and $p = q(p_0, \boldsymbol{p})$ yields

$$q + p = q(q_0 + p_0, \boldsymbol{q} + \boldsymbol{p})$$

The product of imaginary numbers satisfies the following fundamental rules introduced by Hamilton:

$$\begin{aligned} \boldsymbol{i}^2 &= \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \\ \boldsymbol{ij} &= \boldsymbol{k} = -\boldsymbol{ji} \\ \boldsymbol{jk} &= \boldsymbol{i} = -\boldsymbol{kj} \\ \boldsymbol{ki} &= \boldsymbol{j} = -\boldsymbol{ik} \end{aligned}$$

Note that these rules make quaternions non-commutative under multiplication. With these rules, the product of two quaternion is defined as

$$\begin{aligned} pq &= (p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k})(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \\ &= p_0q_0 - \boldsymbol{p}\cdot\boldsymbol{q} + p_0\boldsymbol{q} + q_0\boldsymbol{p} + \boldsymbol{p}\times\boldsymbol{q} \end{aligned}$$

One can see that the cross-product term in the above equation reflects the non-commutative property.

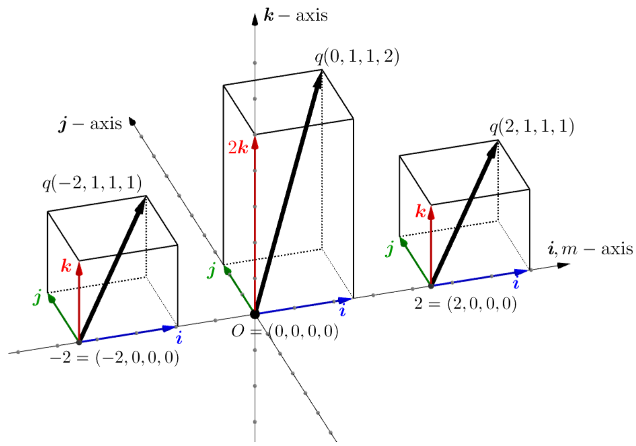


FIGURE 1. Representation of quaternions: $q(-2, 1, 1, 1)$, $q(2, 1, 1, 0)$, and a pure quaternion $q(0, 1, 1, 2)$. The m-axis is for the scalar part, and overlaps the $\boldsymbol{i}$-axis.

The conjugate of a quaternion is similar to that of a complex number; e.g., the conjugate q^* of q is defined as

$$\begin{aligned} q^* &= q_0 - \boldsymbol{q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k} \\ (qp)^* &= p^*q^*. \end{aligned}$$

Similar to the norm of 3-dimensional vectors, the norm $\|q\|$ of a quaternion q is defined as

$$\begin{aligned} \|q\| &= \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = \sqrt{q_0^2 + \|\boldsymbol{q}\|^2} = \sqrt{qq^*} \\ \|qp\| &= \|q\|\,\|p\| \end{aligned}$$

and a quaternion is called a unit quaternion if its norm is 1.

III. GEOMETRIC INTERPRETATION OF QUATERNIONS

Because quaternions are 4-tuples, visualizing them requires the construction of a 4-dimensional geometric model. However, humans visualize the world in three-dimensions, so four orthogonal axes cannot be drawn explicitly. As an alternate approach, this section provides a model by focusing on the fact that a quaternion can be divided into a scalar and a vector. The scalar component can be interpreted as an additional property against 3-dimensions. For example, the mass points model in [6] presents quaternions by considering the scalar component as mass. In this paper, it is more generally interpreted as a 1-dimensional location of the vector space. This is quite reasonable from a geometric perspective because every point in $\mathbb{R}$ should have its own $\mathbb{R}^3$ to span $\mathbb{R}^4$ ($\mathbb{R}^4 = \mathbb{R}^1 \otimes \mathbb{R}^3$). Hereafter, quaternion $q = (q_0, q_1, q_2, q_3)$ is interpreted as follows, "The origin of the vector space is located at q_0 on the axis of the scalar part, and the vector $\boldsymbol{q}(q_1, q_2, q_3)$ is in this space." As per this interpretation, the corresponding geometric model is constructed by first drawing an axis to represent the scalar part and then locating the origin of the 3-dimensional vector space at the zero point on the axis. Then, the origin of the 4-dimensional quaternion space is denoted by (0, 0, 0, 0), and $(q_0, 0, 0, 0)$ is obtained by translating it by a distance q_0 along the axis. As an example, Fig. 1 depicts

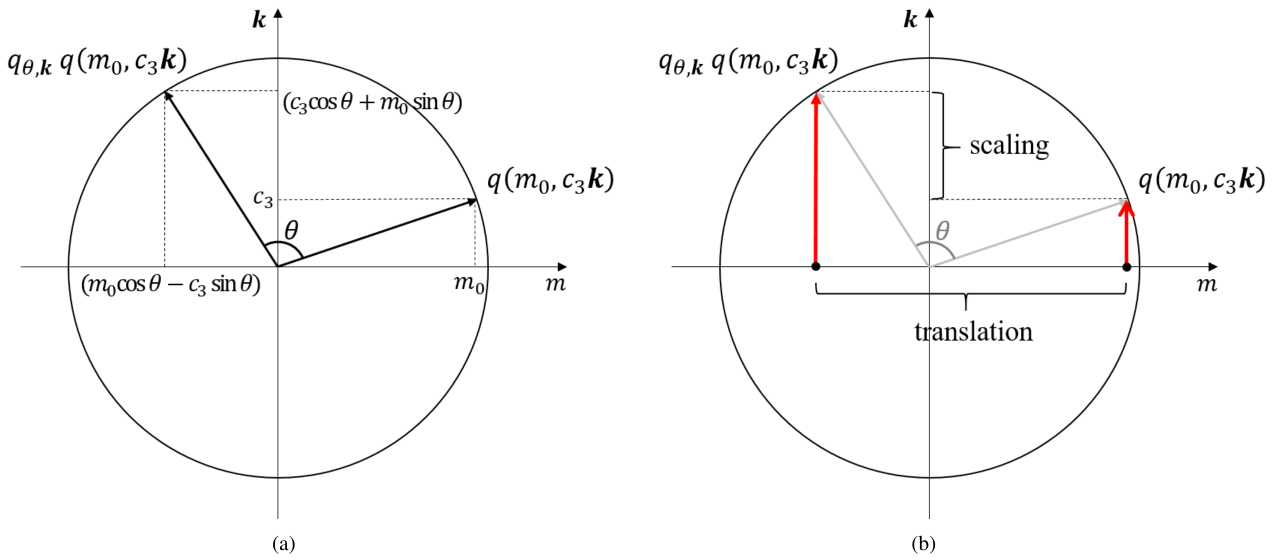


FIGURE 2. Geometric interpretations of Part 1. (a) Represents a typical interpretation of complex multiplication (rotation) considering the scalar part as the 1-dimensional location of the vector space. (b) Represents a modified interpretation of Part 1 (translation and scaling of the vector space).

three quaternions including a pure quaternion. The fourth axis m in the model is provided for a translation axis of 3-dimensions. Although the translation axis can be chosen for any direction, to ensure consistent visualization of quaternions it must be fixed once the direction is selected. This paper sets the translation axis as the i-axis, and accordingly draws the model by overlapping the scalar part axis with the i-axis.

IV. GEOMETRIC INTERPRETATION OF QUATERNION MULTIPLICATION

Most researchers in engineering fields and students who take courses that cover quaternions have encountered complex numbers and their operations pertaining to 2-dimensional rotation. Thus, understanding quaternion multiplication by maintaining a connection to the prior knowledge on complex numbers is helpful. Therefore, this section first considers how a unit complex number acts on a quaternion when they are multiplied together. Notice that this is a special case of quaternion multiplication because complex numbers can be represented by a quaternion. Next, we extend the results to more-general cases where a quaternion is multiplied by a unit quaternion.

A. COMPLEX MULTIPLICATION TO A QUATERNION

A unit complex number acts as a rotation operator over 2-dimensional space and can be written as $\cos\theta + \boldsymbol{k}\sin\theta$, where θ is the rotation angle and $\boldsymbol{k}$ is a unit vector represented by an imaginary number. Note that the complex number can be represented by a unit quaternion as follows

$$q_{\theta,\boldsymbol{k}} = q(\cos\theta, \boldsymbol{k}\sin\theta)$$

Then, as a special case of quaternion multiplication, the left multiplication of the 2-dimensional rotation operator $q_{\theta,\boldsymbol{k}}$ to a quaternion $q(m_0, \boldsymbol{v})$ can be divided into two parts as follows

$$\begin{aligned} q_{\theta,\boldsymbol{k}} q(m_0, \boldsymbol{v}) &= q_{\theta,\boldsymbol{k}}\{q(m_0, c_3\boldsymbol{k}) + q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j})\} \\ &= \underbrace{q_{\theta,\boldsymbol{k}} q(m_0, c_3\boldsymbol{k})}_{\text{Part 1}} + \underbrace{q_{\theta,\boldsymbol{k}} q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j})}_{\text{Part 2}} \end{aligned}$$

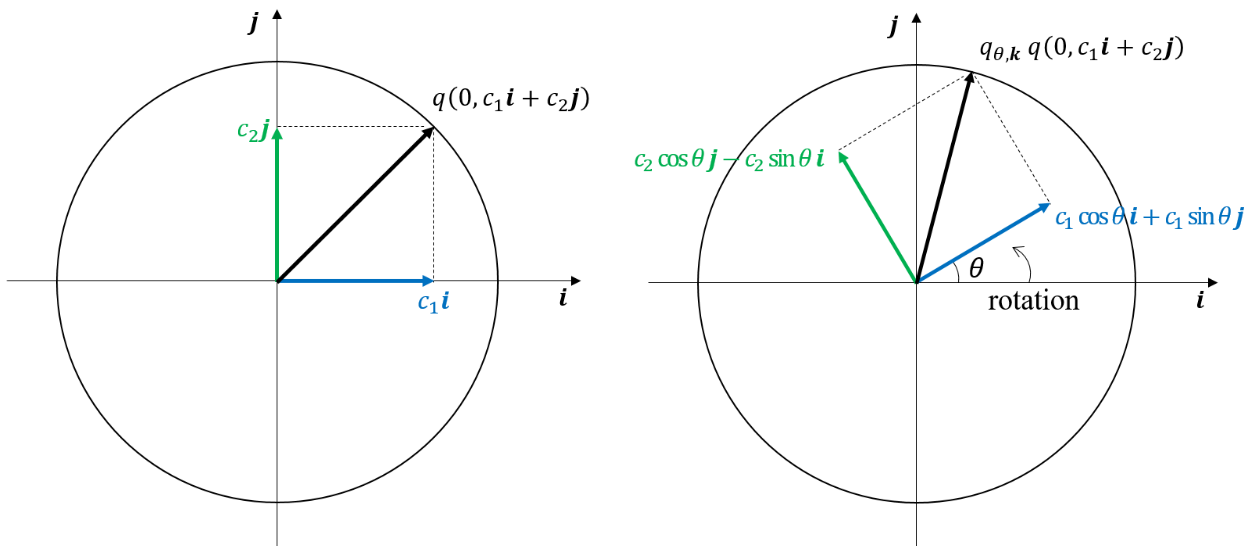
where $\boldsymbol{v} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$ is a 3-dimensional vector. Note that, Part 1 is a complex multiplication and computed as follows

$$\begin{aligned} q_{\theta,\boldsymbol{k}} q(m_0, c_3\boldsymbol{k}) &= (cos\theta + \boldsymbol{k}sin\theta)(m_0 + c_3\boldsymbol{k}) \\ &= (m_0 cos\theta - c_3 sin\theta) + (c_3 cos\theta + m_0 sin\theta)\boldsymbol{k} \end{aligned}$$

From the typical interpretation of complex multiplication, one can easily verify that the geometric meaning is counterclockwise rotation of the vector $m_0 + c_3\boldsymbol{k}$ in the $m\boldsymbol{k}$-plane (see Fig. 2a). Now, let the scalar part denote the location of the 3-dimensional vector space as described in Section III. Then, Part 1 can be interpreted as the joint effect of translation and scaling of the vector space (see Fig. 2b). More precisely, it can be said that a unit complex number $q_{\theta,\boldsymbol{k}}$ translates the vector space of $q(m_0, \boldsymbol{v})$ with its reduction (or expansion) in the direction of $\boldsymbol{k}$ by Part 1 (see Fig. 4a). Meanwhile, Part 2 can be refactored as follows

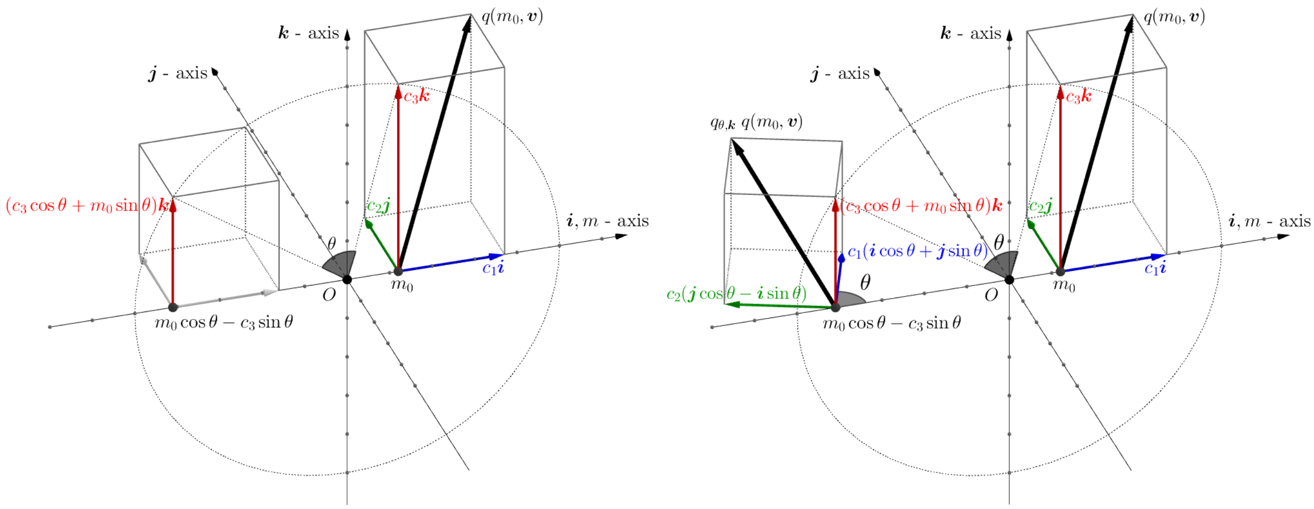
$$\begin{aligned} q_{\theta,\boldsymbol{k}} q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j}) &= c_1 q_{\theta,\boldsymbol{k}}\boldsymbol{i} + c_2 q_{\theta,\boldsymbol{k}}\boldsymbol{j} \\ &= c_1 \underbrace{(\boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta)}_{\text{rotation of } \boldsymbol{i} \text{ axis}} \\ &\quad + c_2 \underbrace{(\boldsymbol{j}\cos\theta - \boldsymbol{i}\sin\theta)}_{\text{rotation of } \boldsymbol{j} \text{ axis}} \end{aligned}$$

Thus, Part 2 implements counterclockwise rotation of the vector space of $q(m_0, \boldsymbol{v})$ around the rotation axis $\boldsymbol{k}$ by θ (see Fig. 3). Now, the sum of Part 1 and 2 reveals that a unit complex number causes translation, scaling, and rotation of 3-dimensional vector space when multiplied to a quaternion on the left-hand side (see Fig. 4b).



(a) (b)

FIGURE 3. Geometric interpretation of Part 2. (a) Represents a reference vector and (b) represents the rotational effect of Part 2.



(a) (b)

FIGURE 4. Quaternion multiplication is divided into two parts: ***translation and scaling*** (Part 1) and ***rotation*** (Part 2) of the vector space. (a) Represents translation and scaling of the vector space (Part 1), and (b) represents the sum of two parts (Part 1 + Part 2).

One of the main benefits of geometric interpretations of quaternion multiplication is that it can be used to derive results without their direct computation. The proposed model is most beneficial in this regard. To be precise, Fig. 5 depicts variations of $q_{\theta,\boldsymbol{k}}q(0, \boldsymbol{v})$ for changes in θ.[1]

Noting that quaternion multiplication is non-commutative, one can expect that the above results would change for the right multiplication. The following propositions formalize the difference between the left and right multiplication. From the multiplication rule of quaternion, one can easily verify that the following propositions hold; thus, we omit the proof.

[1]This paper has a supplementary video file which shows the dynamic variation. This will be available at http://ieeexplore.ieee.org.

Proposition 1: For $q_{\theta,\boldsymbol{k}} = \cos\theta + \boldsymbol{k}\sin\theta$ and $m_0, c_3 \in \mathbb{R}$,

$$\text{(a). } q(m_0, c_3\boldsymbol{k})q_{\theta,\boldsymbol{k}} = q_{\theta,\boldsymbol{k}}q(m_0, c_3\boldsymbol{k})$$

$$\text{(b). } \underbrace{q(m_0, c_3\boldsymbol{k})}_{\text{No effect}} = \underbrace{q_{\theta,\boldsymbol{k}}q(m_0, c_3\boldsymbol{k})}_{\text{Part 1}}\, q^*{}_{\theta,\boldsymbol{k}} = q^*{}_{\theta,\boldsymbol{k}}\, \underbrace{q_{\theta,\boldsymbol{k}}q(m_0, c_3\boldsymbol{k})}_{\text{Part 1}}$$

Proposition 2: For $q_{\theta,\boldsymbol{k}} = \cos\theta + \boldsymbol{k}\sin\theta$ and $c_1, c_2 \in \mathbb{R}$,

$$\text{(a). } q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j})q_{\theta,\boldsymbol{k}} = q^*{}_{\theta,\boldsymbol{k}}q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j})$$

$$\text{(b). } \underbrace{q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j})}_{\text{No effect}} = \underbrace{q_{\theta,\boldsymbol{k}}q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j})}_{\text{Part 2}}\, q_{\theta,\boldsymbol{k}} = q^*{}_{\theta,\boldsymbol{k}}\, \underbrace{q_{\theta,\boldsymbol{k}}q(0, c_1\boldsymbol{i} + c_2\boldsymbol{j})}_{\text{Part 2}}$$

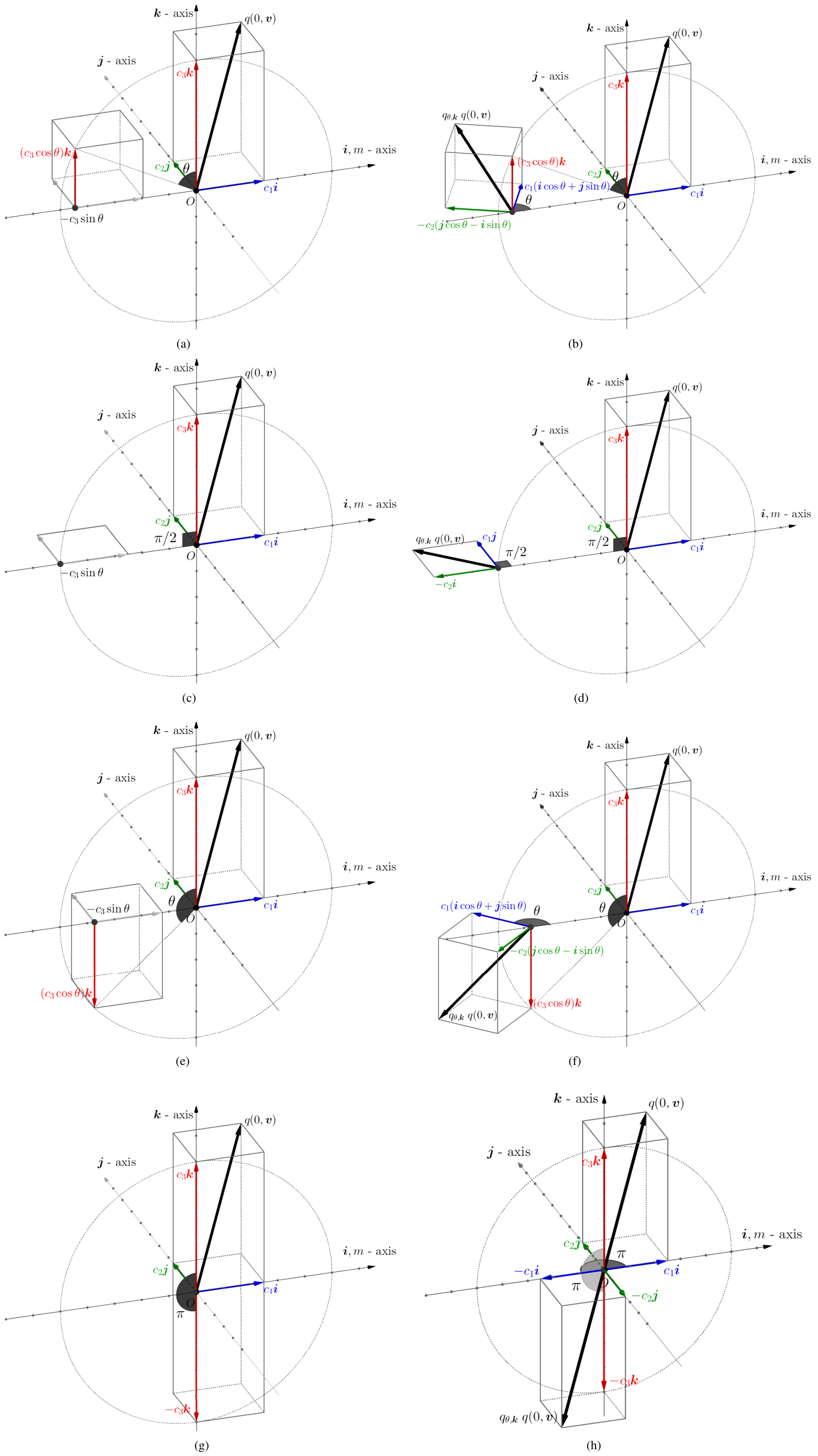


FIGURE 5. Left-hand-side multiplication of a unit quaternion $q_{\theta,k}$ to a pure quaternion $q(0, v)$. (a) Part 1 with $0 < \theta < \pi/2$. (b) Part 1 + Part 2 with $0 < \theta < \pi/2$. (c) Part 1 with $\theta = \pi/2$. (d) Part 1 + Part 2 with $\theta = \pi/2$. (e) Part 1 with $\pi/2 < \theta < \pi$. (f) Part 1 + Part 2 with $\pi/2 < \theta < \pi$. (g) Part 1 with $\theta = \pi$. (h) Part 1 + Part 2 with $\theta = \pi$.

Proposition 1(a) implies that Part 1 is commutative, and 1(b) implies that the effects of Part 1 (translation and scaling of the vector space) can be cancelled out by multiplying the conjugate $q^*_{\theta,k}$ on either the left-hand-side or the right-hand-side. Proposition 2(a) implies that Part 2 is non-commutative and that the right multiplication causes clockwise rotation, whereas the left multiplication causes counterclockwise rotation. Proposition 2(b) implies that unlike Part 1 the effect of Part 2 (rotation of the vector space) can be cancelled out by multiplying the conjugate ${q^*}_{\theta,\boldsymbol{k}}$ on the left-hand-side or by multiplying the rotation operator $q_{\theta,\boldsymbol{k}}$ once again on the right-hand-side.

From the above two propositions, we can discover that one cancelling out mechanism requires the conjugate while the other does not. Therefore, this asymmetric property can be exploited to leave only desirable effects related to Part 1 or Part 2. This property will be used in Section V when deriving quaternion rotation formulae.

B. QUATERNION MULTIPLICATION TO A QUATERNION

The explanation that has been given up to now can be extended to more-general cases in which a quaternion is multiplied by a unit quaternion. The idea behind this generalization is the fact that any unit quaternion can be regarded as a unit complex number by the following lemma:

Lemma 1: Given arbitrary three orthonormal vectors spanning $\mathbb{R}^3$, denoted by $\boldsymbol{q}$, $\boldsymbol{p}$, $\boldsymbol{n}(=\boldsymbol{q}\times\boldsymbol{p})$, each of them can be regarded as an imaginary number $\boldsymbol{k}'$, $\boldsymbol{i}'$, $\boldsymbol{j}'$, respectively.

Proof: Note that $\boldsymbol{i}^2 = q(0,\boldsymbol{i})q(0,\boldsymbol{i})$. Then,

$$\begin{aligned}\boldsymbol{q}^2 &= q(0,\boldsymbol{q})q(0,\boldsymbol{q}) = -\boldsymbol{q}\cdot\boldsymbol{q}+\boldsymbol{q}\times\boldsymbol{q} = -1+0 = -1\\ \boldsymbol{qp} &= q(0,\boldsymbol{q})q(0,\boldsymbol{p}) = -\boldsymbol{q}\cdot\boldsymbol{p}+\boldsymbol{q}\times\boldsymbol{p} = \boldsymbol{n}\\ \boldsymbol{pq} &= q(0,\boldsymbol{p})q(0,\boldsymbol{q}) = -\boldsymbol{p}\cdot\boldsymbol{q}+\boldsymbol{p}\times\boldsymbol{q} = -\boldsymbol{n}\end{aligned}$$

Similarly, the following hold:

$$\begin{aligned}\boldsymbol{p}^2 &= \boldsymbol{n}^2 = \boldsymbol{q}^2 = \boldsymbol{pnq} = -1\\ \boldsymbol{pn} &= \boldsymbol{q} = -\boldsymbol{np}\\ \boldsymbol{nq} &= \boldsymbol{p} = -\boldsymbol{qn}\\ \boldsymbol{qp} &= \boldsymbol{n} = -\boldsymbol{pq}\end{aligned}$$

The above equations show that $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{n}$ satisfy the property of imaginary numbers and the multiplication rule of quaternion. Therefore, each of them can be regarded as an imaginary number $\boldsymbol{k}', \boldsymbol{i}', \boldsymbol{j}'$, respectively. ■

By Lemma 1, a unit quaternion can be regarded as a unit complex number as $q_{\theta,\boldsymbol{q}} = q_{\theta,\boldsymbol{k}'}$. Now, noting that

$$\begin{aligned}q(m_0,\boldsymbol{v}) &= q(m_0, (\boldsymbol{v}\cdot\boldsymbol{p})\boldsymbol{p}+(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}+(\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q})\\ &= q(m_0, (\boldsymbol{v}\cdot\boldsymbol{i}')\boldsymbol{i}'+(\boldsymbol{v}\cdot\boldsymbol{j}')\boldsymbol{j}'+(\boldsymbol{v}\cdot\boldsymbol{k}')\boldsymbol{k}')\end{aligned}$$

with orthonormal vectors $\boldsymbol{p}, \boldsymbol{n}, \boldsymbol{q}$, we can refactor the quaternion multiplication $q_{\theta,\boldsymbol{q}}q(m_0,\boldsymbol{v})$ as follows

$$\begin{aligned}q_{\theta,\boldsymbol{q}}q(m_0,\boldsymbol{v}) &= q_{\theta,\boldsymbol{q}}\{q(m_0,(\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q})+q(0,(\boldsymbol{v}\cdot\boldsymbol{p})\boldsymbol{p}+(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n})\}\\ &= \underbrace{q_{\theta,\boldsymbol{k}'}q(m_0,(\boldsymbol{v}\cdot\boldsymbol{k}')\boldsymbol{k}')}_{\text{Part 1}}\\ &\quad + \underbrace{q_{\theta,\boldsymbol{k}'}q(0,(\boldsymbol{v}\cdot\boldsymbol{i}')\boldsymbol{i}'+(\boldsymbol{v}\cdot\boldsymbol{j}')\boldsymbol{j}')}_{\text{Part 2}}\end{aligned}$$

Then, we can apply the interpretation and the visualization given in the previous special case to the general quaternion multiplication $q_{\theta,\boldsymbol{q}}q(m_0,\boldsymbol{v})$: Part 1 implies translation of the vector space of $q(m_0,\boldsymbol{v})$ with its reduction (or expansion) in the direction of $\boldsymbol{q}$, and Part 2 implies rotation of the vector space around the rotation axis $\boldsymbol{q}$ by θ. Thus, the geometric meaning of multiplying a unit quaternion can be generally interpreted as translation, scaling, and rotation of 3-dimensional vector space.

V. GEOMETRIC DERIVATION OF ROTATION FORMULA

As discussed in Section IV, rotation of 3-dimensional vector space can be implemented by Part 2 of multiplying a unit quaternion. However, this process simultaneously generates an undesirable effect, which is translation and scaling of the vector space caused by Part 1. Therefore, to implement 3-dimensional rotation by quaternion multiplication, this undesirable effect must be eliminated to obtain a magnitude-invariant rotation. Then, the rotation formula utilizing quaternion multiplication can be derived in many ways according to the compensation schemes. In this section, we introduce two compensation schemes and derive three kinds of quaternion rotation formula.

To begin with, consider the following simple rotation problem:

Problem 1 (Simple 3D-Rotation): Find the rotated vector $\boldsymbol{v}'$ when $\boldsymbol{v} = c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k}$ is rotated counterclockwise by θ around the rotation axis $\boldsymbol{k}$.

In the first method, to only obtain the rotation effect from the quaternion multiplication, one can originally exclude the effect of Part 1 (translation and scaling of the vector space) by subtracting and preserving the rotation axis component $c_3\boldsymbol{k}$ from the object vector $\boldsymbol{v}$. Then, noting that $\boldsymbol{v}' = q(0,\boldsymbol{v}')$, we can formulate two rotation formulae:

Formula 1: $\boldsymbol{v}' = q_{\theta,\boldsymbol{k}}\, q(0,\boldsymbol{v}-c_3\boldsymbol{k})+c_3\boldsymbol{k}$

Formula 2: $\boldsymbol{v}' = q_{\theta,\boldsymbol{k}}\, q(c_3\sin\theta, \boldsymbol{v}-(1-\cos\theta)c_3\boldsymbol{k})$

Formula 1 is obtained by directly formulating the compensation scheme, and Formula 2 is obtained from Formula 1 as follows

$$\begin{aligned}q(0,\boldsymbol{v}') &= q_{\theta,\boldsymbol{k}}\, q(0,\boldsymbol{v}-c_3\boldsymbol{k})+c_3\boldsymbol{k}\\ &= q_{\theta,\boldsymbol{k}}\, q(0,\boldsymbol{v}-c_3\boldsymbol{k})+c_3q_{\pi/2,\boldsymbol{k}}\\ &= q_{\theta,\boldsymbol{k}}\, q(0,\boldsymbol{v}-c_3\boldsymbol{k})+c_3q_{\theta,\boldsymbol{k}}q_{\pi/2-\theta,\boldsymbol{k}}\\ &= q_{\theta,\boldsymbol{k}}(q(0,\boldsymbol{v}-c_3\boldsymbol{k})+c_3q_{\pi/2-\theta,\boldsymbol{k}})\\ &= q_{\theta,\boldsymbol{k}}\{q(0,\boldsymbol{v}-c_3\boldsymbol{k})+c_3q(\sin\theta,\boldsymbol{k}\cos\theta)\}\\ &= q_{\theta,\boldsymbol{k}}\, q(c_3\sin\theta,\boldsymbol{v}-(1-\cos\theta)c_3\boldsymbol{k})\end{aligned}$$

In the second method, we use the asymmetric property of the cancelling out mechanism, which is discussed in Section IV. From Proposition 1 and 2, note that the right multiplication of the conjugate quaternion to $q_{\theta,\boldsymbol{k}}q(0,\boldsymbol{v})$ (i.e., $q_{\theta,\boldsymbol{k}}q(0,\boldsymbol{v}){q^*}_{\theta,\boldsymbol{k}}$) cancels out the effect of Part 1 and preserves the effect of Part 2. That is, $q_{\theta,\boldsymbol{k}}q(0,\boldsymbol{v}){q^*}_{\theta,\boldsymbol{k}}$ implies the rotation of the vector space of q(0,$\boldsymbol{v}$) by 2θ around the rotation axis $\boldsymbol{k}$. Then, by replacing θ with $\theta/2$, we can derive the third rotation formula, which is the sandwiching multiplication form:

Formula 3: $q(0,\boldsymbol{v}') = q_{\theta/2,\boldsymbol{k}}\, q(0,\boldsymbol{v})\, {q^*}_{\theta/2,\boldsymbol{k}}$

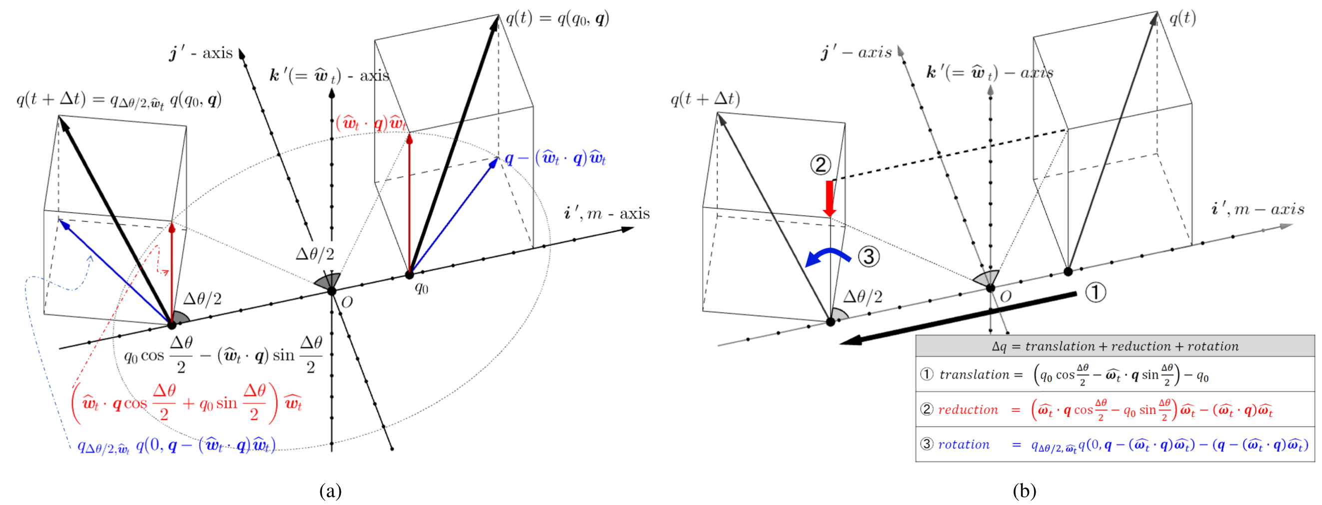


FIGURE 6. (a) $q(t + \Delta t)$ is obtained after some extra rotation of $q(t)$ during Δt. The corresponding rotation axis is equal to the instantaneous angular velocity $\widehat{\omega}_t = \omega_t/\|\omega_t\|$. (b) $\Delta q = q(t + \Delta t) - q(t)$ is divided into three parts as $\Delta q = translation + reduction + rotation$.

Now, consider the following generalized problem:

Problem 2 (General 3D-Rotation): Find the rotated vector $\boldsymbol{v}'$ *when* $\boldsymbol{v} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$ *is rotated counterclockwise by* θ *around the rotation axis* $\boldsymbol{q}$*, where* $\|\boldsymbol{q}\| = 1$.

By using Lemma 1, Formula 1-3 can be generalized as the following Theorem 1, which covers all kinds of rotation axis $\boldsymbol{q}$:

Theorem 1: Let $\boldsymbol{v}'$ *be the rotated vector of* $\boldsymbol{v}$ *around the rotation axis* $\boldsymbol{q}$ *by* θ*, where* $\|\boldsymbol{q}\| = 1$*. Then,*

$$\text{(a). } \boldsymbol{v}' = q_{\theta,\boldsymbol{q}}q(0, \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q}) + (\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q}$$
$$\text{(b). } \boldsymbol{v}' = q_{\theta,\boldsymbol{q}}q((\boldsymbol{v}\cdot\boldsymbol{q})\sin\theta, \boldsymbol{v} - (1-\cos\theta)(\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q})$$
$$\text{(c). } \boldsymbol{v}' = q_{\theta/2,\boldsymbol{q}}q(0, \boldsymbol{v}){q^*}_{\theta/2,\boldsymbol{q}}$$

Proof: (a): Let $\boldsymbol{q}, \boldsymbol{p}$, and $\boldsymbol{n}(=\boldsymbol{q}\times\boldsymbol{p})$ be orthonormal vectors spanning $\mathbb{R}^3$. Then, $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q} + (\boldsymbol{v}\cdot\boldsymbol{p})\boldsymbol{p} + (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$, and by Lemma 1, $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{k}')\boldsymbol{k}' + (\boldsymbol{v}\cdot\boldsymbol{i}')\boldsymbol{i}' + (\boldsymbol{v}\cdot\boldsymbol{j}')\boldsymbol{j}'$, where $\boldsymbol{q} = \boldsymbol{k}'$, $\boldsymbol{p} = \boldsymbol{i}'$, and $\boldsymbol{n} = \boldsymbol{j}'$. Now by Formula 1,

$$\begin{aligned} &q_{\theta,\boldsymbol{k}'}\, q(0, \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{k}')\boldsymbol{k}') + (\boldsymbol{v}\cdot\boldsymbol{k}')\boldsymbol{k}' \\ &\quad = q_{\theta,\boldsymbol{q}}\, q(0, \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q}) + (\boldsymbol{v}\cdot\boldsymbol{q})\boldsymbol{q} \end{aligned}$$

rotates the vector $\boldsymbol{v}$ around the rotation axis $\boldsymbol{k}'(=\boldsymbol{q})$ by θ. (b) and (c): Similar to (a), c_3 and $\boldsymbol{k}$ in Formula 2-3 can be replaced with $(\boldsymbol{v}\cdot\boldsymbol{q})$ and $\boldsymbol{q}$, respectively. ■

Theorem 1(c) is usually called *the sandwiching formula*, and $q_{\theta/2,\boldsymbol{q}}$ is called *a quaternion rotation operator*. Although all of the three formulae implement the same 3-dimensional rotation, Theorem 1(c) has been widely used because it is most compact and convenient to represent sequential rotations as follows

$$\begin{aligned} q(0, \boldsymbol{v}') &= q_{\theta_2/2,\boldsymbol{q}_2}q_{\theta_1/2,\boldsymbol{q}_1}q(0, \boldsymbol{v}){q^*}_{\theta_1/2,\boldsymbol{q}_1}{q^*}_{\theta_2/2,\boldsymbol{q}_2} \\ &= q_{\theta_3/2,\boldsymbol{q}_3}q(0, \boldsymbol{v}){q^*}_{\theta_3/2,\boldsymbol{q}_3} \end{aligned}$$

where $q_{\theta_3/2,\boldsymbol{q}_3} = q_{\theta_2/2,\boldsymbol{q}_2}q_{\theta_1/2,\boldsymbol{q}_1}$. However, when it comes to a single rotation, using Theorem 1(a) or 1(b) is more convenient because it requires fewer computations than the sandwiching formula.

VI. GEOMETRIC DERIVATION OF DERIVATIVE FORMULA

In this section, we derive the derivative formula of quaternion based on the physical meaning of quaternion multiplication.

The quaternion rotation operator $q_{\theta/2,\boldsymbol{q}}$ implies a unique rotation, thus it can be used to represent the orientation of an object in space. In this case, the infinitesimal change in orientation can be represented by rotation, and the corresponding rotation axis can be determined by the instantaneous angular velocity vector. Therefore, if $q(t)$ is a quaternion rotation operator that represents the orientation of an object at time t, then it satisfies $q(t + \Delta t) = q_{\Delta\theta/2,\widehat{\boldsymbol{\omega}}_t}q(t)$, where $\boldsymbol{\omega_t}$ is the instantaneous angular velocity vector $\widehat{\boldsymbol{\omega}}_t = \boldsymbol{\omega_t}/\|\boldsymbol{\omega_t}\|$ at time t and $\Delta\theta = \|\boldsymbol{\omega_t}\|\Delta t$. Then, the instantaneous change Δq at $q(t)$ is divided into three parts (see Fig. 6) as

$$\Delta q = translation + scaling + rotation$$

If $q(t) = (q_0, \boldsymbol{q})$, then the three parts are defined as

$$\begin{aligned} translation &= \left(q_0\cos\frac{\Delta\theta}{2} - \widehat{\boldsymbol{\omega}}_t\cdot\boldsymbol{q}\sin\frac{\Delta\theta}{2}\right) - q_0 \\ scaling &= \left(\widehat{\boldsymbol{\omega}}_t\cdot\boldsymbol{q}\cos\frac{\Delta\theta}{2} + q_0\sin\frac{\Delta\theta}{2}\right)\widehat{\boldsymbol{\omega}}_t - (\widehat{\boldsymbol{\omega}}_t\cdot\boldsymbol{q})\widehat{\boldsymbol{\omega}}_t \\ rotation &= q_{\Delta\theta/2,\widehat{\boldsymbol{\omega}}_t}q(0, \boldsymbol{q} - (\widehat{\boldsymbol{\omega}}_t\cdot\boldsymbol{q})\widehat{\boldsymbol{\omega}}_t) \\ &\quad - q(0, \boldsymbol{q} - (\widehat{\boldsymbol{\omega}}_t\cdot\boldsymbol{q})\widehat{\boldsymbol{\omega}}_t) \end{aligned}$$

and the following holds:

$$\begin{aligned} \lim_{\Delta t\to 0}(translation/\Delta t) &= -(\boldsymbol{\omega_t}\cdot\boldsymbol{q})/2 \\ \lim_{\Delta t\to 0}(scaling/\Delta t) &= (q_0\boldsymbol{\omega_t})/2 \\ \lim_{\Delta t\to 0}(rotation/\Delta t) &= (\boldsymbol{\omega_t}\times\boldsymbol{q})/2 \end{aligned}$$

Finally, the sum of the above limits yields the derivative of

$q(t)$ as follows

$$
\begin{aligned}
dq(t)/dt &= \lim_{\Delta t \to 0} (\Delta q / \Delta t) \\
&= \lim_{\Delta t \to 0} \frac{(translation + scaling + rotation)}{\Delta t} \\
&= \left(-\boldsymbol{\omega_t} \cdot \boldsymbol{q} + q_0 \boldsymbol{\omega_t} + \boldsymbol{\omega_t} \times \boldsymbol{q}\right)/2 \\
&= q(0, \boldsymbol{\omega_t}) q(t)/2
\end{aligned}
$$

VII. CONCLUSION

This paper presented a new geometric approach to quaternion multiplication by expressing the quaternion space as a 3-dimensional space that can move along a linear axis. By considering the axis of the scalar part as a 1-dimensional translation axis of 3-dimensional vector space, the operation of multiplying a quaternion is interpreted as translation, scaling, and rotation of the vector space. Through this interpretation, quaternion rotation formulae and the derivative of quaternions are demonstrated. The value of the proposed geometric approach lies in its practicality for providing a visual and easy-to-understand description of quaternion multiplication. One can understand the entire quaternion algebra in a visual way using the proposed geometric model.

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