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The dimension of the range of a transient random walk*

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Abstract

We find formulas for the macroscopic Minkowski and Hausdorff dimensions of the range of an arbitrary transient walk in $\mathbb{Z}^d$. This endeavor solves a problem of Barlow and Taylor (1991).

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1 Introduction

Throughout, we choose and fix an integer $d \geqslant 1$, and let $\mathbb{Z}^d$ denote the corresponding d-dimensional integer lattice. Our main object of study is the dimension of the range of arbitrary random walks with state space a subset of $\mathbb{Z}^d$. With this aim in mind, let $X := \{X_n\}_{n=0}^\infty$ denote a random walk on $\mathbb{Z}^d$, started at some deterministic point $X_0 := a \in \mathbb{Z}^d$, and denote its range by $\mathcal{R}_X := \cup_{n=0}^\infty \{X_n\}$. Our goal is to give an answer to the following question of Barlow and Taylor [2, Problem, p. 145]:

$$\text{What is } \mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X)? \tag{1.1}$$

Here, $\mathrm{Dim}_{\mathrm{H}}(G)$ denotes the macroscopic Hausdorff dimension of a set $G \subset \mathbb{R}^d$, as was defined in [1, 2]. We will recall the formal definition and first properties of $\mathrm{Dim}_{\mathrm{H}}$ in §2.1 below. In informal terms, $\mathrm{Dim}_{\mathrm{H}}(G)$ measures the "local dimension of G at infinity," and

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describes the large-scale geometry of G in a manner that is similar to the way that the ordinary, microscopic, Hausdorff dimension of G describes the small-scale geometry of G.

Barlow and Taylor [2] pose (1.1) only for transient random walks. We briefly explain why answering (1.1) in the case of a recurrent walk is straight-forward. Here, "recurrent" means "recurrent on its range"; the walk can still avoid significant portions of $\mathbb{Z}^d$. Recall that a point $x \in \mathbb{Z}^d$ is *possible* for X if $P^0\{X_n = x\} > 0$ for some $n \geqslant 1$, where P^a denotes the conditional law of X given that $X_0 = a$, as usual. The collection of all possible points of X is an additive subgroup of $\mathbb{Z}^d$, whence homomorphic to $\mathbb{Z}^k$ for some integer $0 \leqslant k \leqslant 2$. [This is valid by the structure theory of finitely-generated abelian groups and the fact that X is transient provided that the collection of possible points of X is homomorphic to $\mathbb{Z}^\ell$ for $\ell \geqslant 3$.] Therefore, the strong Markov property of X implies that $\mathcal{R}_X$ is homomorphic to $\mathbb{Z}^k$ a.s., and hence we have $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) = \mathrm{Dim}_{\mathrm{H}}(\mathbb{Z}^k) = k$ a.s. [2].

As far as we know, the only existing positive result about (1.1) is due to Barlow and Taylor themselves [2, Cor. 7.9]. In order to describe their result, let g denote the Green function of X. That is,

$$g(a\,,x) := \sum_{n=0}^{\infty} P^a\{X_n = x\}, \qquad x, a \in \mathbb{Z}^d. \tag{1.2}$$

Of course, $g(a\,,x) = g(0\,,x-a)$ as well, and the transience of X is equivalent to the finiteness of the function g on all of $\mathbb{Z}^d \times \mathbb{Z}^d$.

Question (1.1) was in part motivated by the following positive result.

Proposition 1.1 (Barlow and Taylor [2, Corollary 7.9]). *Let $d \geqslant 2$. Suppose there exist constants $\alpha \in (0\,,2]$ and $A, B \in (0\,,\infty)$ such that*

$$A\|x\|^{-d+\alpha} \leqslant g(0\,,x) \leqslant B\|x\|^{-d+\alpha}, \tag{1.3}$$

whenever $x \in \mathbb{Z}^d \setminus \{0\}$. Then,

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) = \alpha \qquad P^0\text{-a.s.}$$

The principal goal of this article is to answer question (1.1) in general. We do this by following a suggestion of Barlow and Taylor and introducing a random-walk "index" that is equal to $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X)$. Moreover, as is tacitly implied in [2], this index is defined solely in terms of the statistical properties of X.

It turns out that our index is related to the notion of a "recurrent set" for X. Recall that a set $F \subseteq \mathbb{Z}^d$ is said to be *recurrent* for X, under P^a, if the random set $X^{-1}(F) := \{n \geqslant 0 : X_n \in F\}$ is unbounded P^a-a.s. This definition makes sense regardless of whether F is random or not.

Because X is transient, a necessary condition for the recurrence of F is that F is unbounded. The following example shows that the converse implication is false: Let X denote the simple symmetric walk on $\mathbb{Z}^3$, and define $F := \cup_{k=1}^{\infty}\{x_k\}$, where $x_k := (0\,,0\,,k^3)$. By the classical local central limit theorem, $P^0\{x_k \in \mathcal{R}_X\} = O(k^{-3})$ as $k \to \infty$. Therefore, Tonelli's theorem implies that

$$E^0\left[\sum_{n=0}^{\infty} \mathbb{1}_F(X_n)\right] = \sum_{k=0}^{\infty} P^0\{x_k \in \mathcal{R}_X\} < \infty,$$

where E^0 denotes the expectation operator for P^0. It follows that F is not recurrent for X, though it is manifestly unbounded. In fact the construction in the example above works as long as $g(0\,,x) \to 0$ as $x \to \infty$; choose a sequence $\{x_k\}_{k=1}^{\infty}$ with $g(0\,,x_k) < 2^{-k}$.

A necessary-and-sufficient condition for the recurrence of a *non-random* set F was found first by Itô and McKean [7] in the case that X is the simple random walk on $\mathbb{Z}^d$ and $d \geqslant 3$. Lamperti [15] discovered a necessary-and-sufficient condition in the case that X belongs to a large family of transient random walks on $\mathbb{Z}^d$. Lamperti's theorem in fact holds for a large family of transient Markov chains X. When X is a general transient random walk on $\mathbb{Z}^d$, or more generally a Markov chain on a countable state space, there is also an exact condition for set recurrence, but that condition is more involved; see Bucy [4] and, more recently, Benjamini et al [3].

All latter works involve various notions of abstract capacity that are borrowed from probabilistic potential theory. Our answer to (1.1) is also stated in terms of a sort of abstract capacity condition, and appears later on as Corollary 5.2 to a master theorem [Theorem 5.1] on the large-scale potential theory of random walks. We have not included our answer to (1.1) in this Introduction since that answer is complicated and its description hinges on first introducing a certain amount of machinery. Still, our answer has a simpler form when the random walk X is sufficiently regular; see Corollary 5.4 for instance.

We conclude the Introduction with an outline of the paper. In §2 we include some of the technical prerequisites to reading this paper. Then, in §3 we develop a macroscopic theory of "fractal percolation" that is the large-scale analogue of the microscopic theory of fractal percolation [17, 20]. Our macroscopic extension of the microscopic theory is not entirely trivial, but will ring familiar to many experts.

In §4 we introduce a forest representation of $\mathbb{Z}^d$ and use it together with the theory of two-parameter processes [9] in order to characterize exactly when $\mathcal{R}_X$ intersects a piece of a macroscopic fractal percolation set. This is the truly-novel part of the present article, and is likely to have other uses particularly in computing the ordinary and/or large-scale dimension of complex random sets.

Finally, in §5 we establish a master theorem on "hitting probabilities"; see Theorem 5.1. Subsequently, we use that master theorem, together with an adaptation of an elegant replica method of Peres [20] to the present setting, in order to compute $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F)$ for every recurrent non-random set F; see Corollary 5.4. Our answer to the original question (1.1) of Barlow and Taylor is obtained by specializing the preceding to $F := \mathbb{Z}^d$.

In the section that follows the above discussion [§6] we derive the following simpler and more elegant almost-sure representation for the macroscopic Minkowski dimension of an arbitrary random walk on $\mathbb{Z}^d$:

$$\mathrm{Dim}_{\mathrm{M}}(\mathcal{R}_X) = \inf\left\{\gamma \in (0\,,d) : \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \frac{g(0\,,x)}{\|x\|^\gamma} < \infty\right\}.$$

In the last section §7 we state a few interesting remaining open problems and related conjectures.

2 Background material

This section introduces the prerequisite material, necessary for later use.

2.1 Macroscopic Hausdorff dimension

Throughout we follow the original notation of Barlow and Taylor [1, 2] by setting

$$\mathcal{V}_k := [-2^k, 2^k)^d, \quad \mathcal{S}_0 := \mathcal{V}_0, \quad \mathcal{S}_{k+1} := \mathcal{V}_{k+1} \setminus \mathcal{V}_k, \tag{2.1}$$

for all $k \geqslant 0$. For every integer n—positive as well as non positive—define D_n to be the collection of all dyadic cubes $Q^{(n)}$ of the form

$$Q^{(n)} := [j_1 2^n\,, (j_1+1)2^n) \times \cdots \times [j_d 2^n\,, (j_d+1)2^n), \tag{2.2}$$

where $j_1, \ldots, j_d \in \mathbb{Z}$ are integers. In the sequel we tacitly will use the fact that the cubes in D_n are actually disjoint.

If a cube $Q^{(n)}$ has the form (2.2), then we say that $j := (j_1, \ldots, j_d)$ is the *southwest corner* of $Q^{(n)}$, and 2^n is the *sidelength* of $Q^{(n)}$.

By $\mathcal{D}$ we mean the collection of all dyadic hypercubes of $\mathbb{Z}^d$; that is,

$$\mathsf{D} := \bigcup_{n=-\infty}^{\infty} \mathsf{D}_n.$$

A special role is played by the collection of all dyadic cubes of sidelength not smaller than 1, which we denoted as

$$\mathsf{D}_{\geqslant 1} := \bigcup_{n=0}^{\infty} \mathsf{D}_n.$$

For every $\alpha \in (0\,,\infty)$ and $A \subseteq \mathbb{R}^d$ define

$$\mathcal{N}_\alpha(A\,,\mathcal{S}_k) := \min \sum_{i=1}^m 2^{\alpha(\ell_i-k-1)} = \min \sum_{i=1}^m [\text{side}(Q_i)/\text{side}(\mathcal{V}_k)]^\alpha, \tag{2.3}$$

where "side" temporarily denotes "sidelength," and the minima are taken over all possible coverings $Q_1, \ldots, Q_m$ of $A \cap \mathcal{S}_k$ such that every $Q_i \subset \mathcal{S}_k$ is a cube of sidelength $2^{\ell_i} \geqslant 1$ whose southwest corner is in $2^{\ell_i}\mathbb{Z}^d$. Note in particular that $Q_1, \ldots, Q_m$ are all elements of $\mathsf{D}_{\geqslant 1}$.

Let us pause momentarily in order to make two small observations.

Remark 2.1. The minimum in (2.3) is taken over a finite set. Therefore, the said minimum is attained for a covering $Q_1, \ldots, Q_m$ of A with respective sidelengths $2^{\ell_1}, \ldots, 2^{\ell_m} \geqslant 1$. We will appeal to the conclusion of this remark tacitly in the sequel.

Remark 2.2. Instead of our $\mathcal{N}_\alpha(A\,,\mathcal{S}_k)$, Barlow and Taylor consider the quantity

$$\mathcal{N}_\alpha(A\,,\mathcal{V}_k) := \min \sum_{i=1}^m 2^{\alpha(\ell_i-k-1)},$$

where now the coverings use dyadic cubes in $\mathcal{V}_k$ rather than dyadic cubes in $\mathcal{S}_k$. Assume for the moment that $A \subseteq \mathcal{S}_k$. Then it follows readily that there exists a real number $c_d > 0$, that depends only on d, such that

$$\mathcal{N}_\alpha(A\,,\mathcal{V}_k) \leqslant \mathcal{N}_\alpha(A\,,\mathcal{S}_k) \leqslant c_d^\alpha \mathcal{N}_\alpha(A\,,\mathcal{V}_k). \tag{2.4}$$

The first inequality above is immediate since every cover that uses $\mathcal{S}_k$-cubes is also cover that uses $\mathcal{V}_k$-cubes. In order to understand the second inequality observe that for every dyadic cube $Q \subseteq \mathcal{V}_k$ there exist at most 2^{2d} dyadic cubes $Q_1, Q_2, \ldots$ in $\mathcal{S}_k$ such that $\text{side}(Q_j) \leqslant \text{side}(Q)$ for all $1 \leqslant j \leqslant 2^{2d}$ and $Q \cap \mathcal{S}_k \subset \cup_{j=1}^{2^{2d}} Q_j$. This is merely an assertion about the geometry of $\mathbb{R}^d$. In any case, it follows from this assertion and Jensen's inequality that

$$[\text{side}(Q)]^\alpha \geqslant \frac{1}{2^{2d\alpha}} \left[\sum_{j=1}^{2^{2d}} \text{side}(Q_j)\right]^\alpha \geqslant \frac{1}{2^{2d\alpha}} \sum_{j=1}^{2^{2d}} [(\text{side}(Q_j)]^\alpha.$$

Therefore, if $Q_1, Q_2, \ldots$ form a minimizing $\mathcal{V}_k$-cover of $A \in \mathcal{S}_k$, then there exist dyadic $\mathcal{S}_k$-cubes $\{Q_{i,j}\}_{1\leqslant j\leqslant 2^{2d}}$, for each $i \geqslant 1$, such that

$$\mathcal{N}_\alpha(A\,,\mathcal{V}_k) = \sum_i \frac{[\text{side}(Q_i)]^\alpha}{2^{k+1}} \geqslant 2^{-2d\alpha} \sum_{i,j} \frac{[\text{side}(Q_{i,j})]^\alpha}{2^{k+1}}.$$

This proves the remaining portion of (2.4) because the cubes in the preceding sum are all in $\mathcal{S}_k$ and they cover A.

We can identitfy $\mathcal{N}_\alpha(A\,,\mathcal{S}_k)$ as the large-scale analogue of the α-dimensional Hausdorff measure of A, restricted to $\mathcal{V}_k$, and scaled so that $\sum_{k=1}^\infty \mathcal{N}_\alpha(A\,,\mathcal{S}_k)$ can serve as a proxy for "macroscopic α-dimensional Hausdorff measure." The latter is not a measure. Still, these kinds of remarks undoubtedly led M. T. Barlow and S. J. Taylor [1, 2] to define the *macroscopic Hausdorff dimension* of A via[1]

$$\mathrm{Dim}_{\mathrm{H}}(A) := \inf\left\{\alpha > 0 : \sum_{k=1}^\infty \mathcal{N}_\alpha(A\,,\mathcal{S}_k) < \infty\right\}. \tag{2.5}$$

We adopt this definition and notation here and throughout.

One can glean many of the properties of $\mathrm{Dim}_{\mathrm{H}}$ from first principles. For instance, it follows readily from the definition of $\mathcal{N}_\alpha$ that $\mathrm{Dim}_{\mathrm{H}}(A) = 0$ if A is a finite set. It is also easy to see that $\mathrm{Dim}_{\mathrm{H}}(A) \leqslant \mathrm{Dim}_{\mathrm{H}}(B)$ whenever $A \subseteq B$. This fact is a consequence of the observation that every covering of B is also a covering of A when $A \subseteq B$. Finally, let us mention that that $\mathrm{Dim}_{\mathrm{H}}(\mathbb{Z}^d) = d$ [1, pp. 2622–2623], and therefore,

$$0 \leqslant \mathrm{Dim}_{\mathrm{H}}(A) \leqslant \mathrm{Dim}_{\mathrm{H}}(B) \leqslant d, \tag{2.6}$$

whenever $A \subseteq B$. The second, seemingly-natural, inequality in (2.6) is one of the novel features of the theory of Barlow and Taylor [1, 2], and does not hold for some of the previously-defined candidates of large-scale dimension in the literature [18, 19].

Let X denote the simple symmetric random walk on $\mathbb{Z}^d$ where $d \geqslant 3$. According to the local central limit theorem, $g(x\,,y) \sim \text{const} \cdot \|x - y\|^{2-d}$ as $\|x - y\| \to \infty$. Therefore, Proposition 1.1 applies, and implies the very appealing fact that the macroscopic Hausdorff dimension of the range of X is a.s. 2.

Barlow and Taylor [2] have proved that macroscopic Hausdorff dimension of the range of transient Brownian motion is also a.s. 2, thus giving further credance to their assertion that $\mathrm{Dim}_{\mathrm{H}}$ is a natural large-scale variation of the classical notion of [microscopic] Hausdorff dimension, nowadays usually denoted by $\dim_{\mathrm{H}}$.

2.2 Recurrent sets for Markov chains

We follow the existing related works on probabilistic potential theory [3, 4, 6, 15], and consider a somewhat more general setting in which our random walk X is replaced by a transient Markov chain, still denoted by X. However, in contrast with prior works we continue to assume that our Markov chain X takes values in the special state space $\mathbb{Z}^d$, for some $d \geqslant 1$, and not in a general countable state space. This assumption is needed for some, but not all, of the ensuing analysis. We make the assumption once and for all in order to avoid studying various special cases, and hence to simplify the exposition. We continue to write $\mathcal{R}_X := \cup_{n=0}^\infty \{X_n\}$ for the range of the Markov chain X.

[1] Barlow and Taylor [1, 2] wrote $\tilde{\nu}_\alpha$ in place of our $\mathcal{N}_\alpha$, and $\dim_{\mathrm{H}}$ in place of our $\mathrm{Dim}_{\mathrm{H}}$. We prefer $\mathcal{N}_\alpha$ as it reminds us that our $\mathcal{N}_\alpha$ is the large-scale analogue of Besicovitch's α-dimensional net measures. And we use for $\mathrm{Dim}_{\mathrm{H}}$ in favor of $\dim_{\mathrm{H}}$ to distinguish between large-scale and ordinary Hausdorff dimension.

As was done for random walks, let P^a denote the conditional law of X, given $X_0 = a$. A random or non-random set $F \subseteq \mathbb{Z}^d$ is said to be *recurrent* for X under P^a when $\mathcal{R}_X \cap F$ is unbounded P^a-a.s.[2]

We are aware of at least two characterizations of non-random recurrent sets for general chains, due to Bucy [4] and Benjamini et al [3]. In order to describe the second characterization, which turns out to be more relevant for our needs, let $M_1(F)$ denote the collection of all probability measures on F, and $c_1(F\,;a)$ the *Martin capacity* of F for the walk started at $a \in \mathbb{Z}^d$ [3]. That is,

$$c_1(F\,;a) := \sup_{\substack{F_0 \subseteq F:\\ F_0 \text{ finite}}} \left[\inf_{\mu \in M_1(F_0)} \sum\sum_{\substack{x,y \in \mathbb{Z}^d:\\ g(a,y)>0}} \frac{g(x\,,y)}{g(a\,,y)} \mu(x)\,\mu(y) \right]^{-1}, \tag{2.7}$$

where $\mu(w) := \mu(\{w\})$ for all $w \in \mathbb{Z}^d$.

Benjamini et al [3] have characterized all recurrent sets for transient Markov chains on countable state spaces. If we apply their result to transient Markov chains on $\mathbb{Z}^d$, then we obtain the following:

Proposition 2.3 (Benjamini, Pemantle, and Peres [3])**.** *Choose and fix some $a \in \mathbb{Z}^d$. A non-random set $F \subseteq \mathbb{Z}^d$ is recurrent for X under P^a if and only if $\inf c_1(G\,;a) > 0$, where the infimum is taken over all cofinite subsets G of F.*

Recall that a set $G \subset \mathbb{Z}^d$ is said to be *cofinite* when $\mathbb{Z}^d \setminus G$ is a bounded set.

The preceding capacity condition for cofinite sets is not so easy to verify in concrete settings. There is an older result, due to Lamperti [15], which contains a more easily-applicable characterization of recurrent sets for “nice” Markov chains. The following is a slightly different formulation that works specifically for transient chains on $\mathbb{Z}^d$. Barlow and Taylor [2, Proposition 8.2] state a special case of it by adapting Lamperti’s method [see Example 2.5 below]. Later on, we derive it as a corollary to a “master theorem” on hitting probabilities of transient chains on $\mathbb{Z}^d$ [Theorem 5.1].

Corollary 2.4 (Lamperti’s test)**.** *Suppose that there exist $a \in \mathbb{Z}^d$ and a finite constant $K > 0$ such that for all $n \geqslant K$ and $m \geqslant n + K$,*

$$\sup_{\substack{x \in \mathcal{S}_n\\ y \in \mathcal{S}_m}} \frac{g(x\,,y)}{g(a\,,y)} + \sup_{\substack{x \in \mathcal{S}_m\\ y \in \mathcal{S}_n}} \frac{g(x\,,y)}{g(a\,,y)} \leqslant K. \tag{2.8}$$

Then, F is recurrent for X under P^a if and only if $\sum_{k=0}^{\infty} c_1(F \cap \mathcal{S}_k; a) = \infty$, where c_1 was defined in (2.7)*.*

Example 2.5. Suppose that X is a random walk that satisfies Condition (1.3) of Proposition 1.1. It readily follows that $g(0\,,y) > 0$ for all $y \neq 0$, and

$$\sum\sum_{x,y \in \mathcal{S}_k} \frac{g(x\,,y)}{g(0\,,y)} \mu(x)\,\mu(y) \asymp 2^{k(d-\alpha)} \sum\sum_{x,y \in \mathcal{S}_k} g(x\,,y)\,\mu(x)\,\mu(y),$$

simultaneously for all integers $k \geqslant 0$ and $\mu \in M_1(F)$. As usual, we write “$f(z) \asymp g(z)$ for all $z \in Z$” to mean that there exists a positive and finite constant C such that $C^{-1}g(z) \leqslant f(z) \leqslant Cg(z)$ for all $z \in Z$. Therefore, $c_1(F \cap \mathcal{S}_k\,;0) \asymp 2^{-k(d-\alpha)} \mathrm{cap}_g(F \cap \mathcal{S}_k)$

[2] One can imagine other variations on this definition. For instance, one could consider F to be recurrent if instead $P^a\{\mathcal{R}_X \cap F \text{ is unbounded}\} > 0$. It should be possible to adjust our methods to study the latter notion of recurrence; see Theorem 5.1 for instance. We are interested mainly in the case where X is a random walk. In that case, the two notions agree. Therefore, we will not puruse these matters further.

for all $k \geqslant 0$, where

$$\mathrm{cap}_g(G) := \left[\inf_{\mu \in M_1(G)} \sum\sum_{x,y \in G} g(x,y)\, \mu(x)\, \mu(y) \right]^{-1}$$

describes the usual random-walk capacity of $G \subset \mathbb{Z}^d$.[3] It is easy to see from (1.3) that the Lamperti-type condition (2.8) also holds in this case. Therefore, Corollary 2.4 tells us that F is recurrent for X under P^0 if and only if $\sum_{k=0}^{\infty} 2^{-k(d-\alpha)} \mathrm{cap}_g(F \cap \mathcal{S}_k) = \infty$. This is Proposition 8.2 of [2].

3 Macroscopic fractal percolation

We temporarily leave the topic of Markov chains and random walks in order to present some basic facts about macroscopic fractal percolation.

Let $k \geqslant 0$ denote a fixed integer, and suppose $\{U(Q) : Q \in \mathsf{D},\, Q \subset \mathcal{V}_k\}$ is a collection of independent random variables, defined on a rich enough probability space $(\Omega\,, \mathcal{F}, \mathrm{P})$, such that each $U(Q)$ is distributed uniformly between 0 and 1. We may define, for all $p \in (0\,,1]$,

$$I_p(Q) := \mathbb{1}_{(0,p)}(U(Q)). \tag{3.1}$$

Then:

(i) $\{I_p(Q) : Q \in \mathsf{D},\, Q \subset \mathcal{V}_k\}$ are i.i.d.;
(ii) $\mathrm{P}\{I_p(Q) = 1\} = p$ and $\mathrm{P}\{I_p(Q) = 0\} = 1 - p$; and
(iii) $I_{p_1}(Q) \leqslant I_{p_2}(Q)$ if $p_1 \leqslant p_2$.

For all integers $k \geqslant 0$ define:

$$\Pi_{p,-1}(\mathcal{V}_k) := \mathcal{V}_k;$$

and then define iteratively for all integers $n \geqslant 0$,

$$\Pi_{p,n}(\mathcal{V}_k) := \{Q \in \mathsf{D}_{k-n} : Q \subseteq \Pi_{p,n-1}(\mathcal{V}_k) \text{ and } I_p(Q) = 1\}\,.$$

In this way, we see for example that $\Pi_{p,0}(\mathcal{V}_k)$ is a random set of cubes with sidelength 2^k which result from the *first step* of a certain branching process; see Figures 1 and 4.

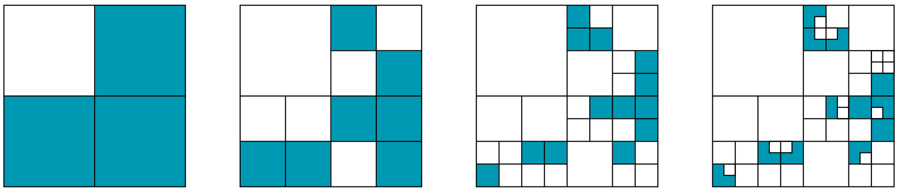

Figure 1: An image of a simulation of stages 2-5 of the construction of fractal percolation in $\mathcal{V}_3$, when $d = 2$ and $p = 1/2$. The first stage is omitted: That stage shows all of $\mathcal{V}_3$ colored in. The sidelength of the white cubes indicates the stage at which the cube was deleted; cubes of smaller sidelength were deleted at a later stage. We stop the process in $\mathcal{V}_3$ after the first 5 stages.

Fractal percolation on $\mathcal{V}_k$ [with parameter $0 < p \leqslant 1$] is the random set

$$\Pi_{p,\infty}(\mathcal{V}_k) := \bigcap_{n=0}^{\infty} \Pi_{p,n}(\mathcal{V}_k). \tag{3.2}$$

[3] Standard last-exit arguments, and/or maximum principle arguments, show that our "cap_g" is the same capacity form as Lamperti's "C" [15] and Barlow and Taylor's "Cap_G" [2]. This fact can be found implicitly in Bucy [4], and might even be older.

One can see that this is the usual construction of Mandelbrot's fractal percolation [17], scaled to take place in the cube $\mathcal{V}_k$. Namely, we may write $\mathcal{V}_k$ as a disjoint union of 2^d elements of D_k; each of those elements is selected independently with probability p and rejected with probability $1-p$. We then write every one of the selected cubes as a disjoint union of 2^d elements of D_{k-1}; each resulting sub-cube is kept/selected or discarded/deselected independently with respective probabilities p and $1-p$; and we continue.

Elementary branching-process theory implies that $\mathrm{P}\{\Pi_{p,\infty}(\mathcal{V}_k) \neq \varnothing\} > 0$ if and only if $p > 2^{-d}$; see also Figures 1, 2, and 4.

Presently, we are interested in performing fractal percolation in $\mathcal{V}_k$ but we will stop the subdivisions after $k+1$ steps. In other words, we are interested in $\Pi_{p,k}(\mathcal{V}_k)$, which is a random, possibly empty, collection of side-one cubes in D_0. Since $I_{p_1}(Q) \leqslant I_{p_2}(Q)$ whenever $p_1 \leqslant p_2$, we see that $\Pi_{p_1,k}(\mathcal{V}_k) \subseteq \Pi_{p_2,k}(\mathcal{V}_k)$ a.s., and hence if $p_1 \leqslant p_2$, then

$$\mathrm{P}\{\Pi_{p_1,k}(\mathcal{V}_k) \cap F \neq \varnothing\} \leqslant \mathrm{P}\{\Pi_{p_2,k}(\mathcal{V}_k) \cap F \neq \varnothing\},$$

for every non-random Borel set $F \subseteq \mathbb{R}^d$.

Now let us construct all of these fractal percolations on the same probability space so that:

(i) $\Pi_{p,k}(\mathcal{V}_k)$ is a fractal percolation in $\mathcal{V}_k$ for every $k \geqslant 0$, as described earlier;

(ii) $\Pi_{p,0}(\mathcal{V}_0), \Pi_{p,1}(\mathcal{V}_1), \cdots$ are independent.

In words, we appeal to the preceding procedure in order to construct the $\Pi_{p,k}(\mathcal{V}_k)$'s simultaneously for all k, using an independent collection of weights $I_p(Q)$'s for each $\mathcal{V}_k$.

By *macroscopic fractal percolation* we mean the random set

$$\Pi_p := \bigcup_{k=0}^{\infty} (\Pi_{p,k}(\mathcal{V}_k) \cap \mathcal{S}_k).$$

Of course, $\Pi_0 = \varnothing$ and $\Pi_1 = \mathbb{R}^d$. Starting from here, we often assume tacitly that $p \in (0\,,1)$ in order to avoid the trivial cases $p = 0$ and $p = 1$. In any case, it is easy to deduce our next result.

Lemma 3.1. *The following are valid:*

1. $\Pi_{p_1} \subseteq \Pi_{p_2}$ *whenever* $p_1 \leqslant p_2$;
2. $\Pi_p \cap \mathcal{S}_0, \Pi_p \cap \mathcal{S}_1, \Pi_p \cap \mathcal{S}_2, \ldots$ *are independent random sets. That is,* $\mathbb{1}_{\Pi_p \cap \mathcal{S}_0}, \mathbb{1}_{\Pi_p \cap \mathcal{S}_1}, \mathbb{1}_{\Pi_p \cap \mathcal{S}_2}, \ldots$ *are independent random variables; and*
3. $\Pi_p \cap \mathcal{S}_k$ *is distributed as* $\Pi_{p,k}(\mathcal{V}_k) \cap \mathcal{S}_k$ *for every integer* $k \geqslant 0$. *In fact, we have the equality of events,*

$$\{\omega\Omega :\ \Pi_p(\omega) \cap \mathcal{S}_k \cap F \neq \varnothing\} = \{\omega \in \Omega :\ \Pi_{p,k}(\mathcal{V}_k)(\omega) \cap \mathcal{S}_k \cap F \neq \varnothing\},$$

valid for all Borel sets $F \subseteq \mathbb{R}^d$.

We will not include the elementary proof.

If $A, B \subseteq \mathbb{R}^d$ are both unbounded, then we say that A is a *recurrent set* for B provided that there exist infinitely many shells $\mathcal{S}_{k_1}, \mathcal{S}_{k_2}, \ldots$ such that $A \cap B \cap \mathcal{S}_{k_n} \neq \varnothing$ for all $n \geqslant 1$. We often write "$A \sim_{(\mathrm{R})} B$" in place of "A is a recurrent set for B." The relation "$\sim_{(\mathrm{R})}$" is reflexive and symmetric for all pairs of unbounded sets.

Lemma 3.2. *If* $F \subset \mathbb{R}^d$ *is non random, then* $\mathrm{P}\{F \sim_{(\mathrm{R})} \Pi_p\} = 0$ *or* 1.

Proof. Let $\zeta_k = 1$ if $F \cap \Pi_p \cap \mathcal{S}_k \neq \varnothing$ and $\zeta_k = 0$ otherwise. Then, the ζ_k's are independent and

$$\mathrm{P}\{F \sim_{(\mathrm{R})} \Pi_p\} = \mathrm{P}\left\{\sum_{k=1}^{\infty} \zeta_k = \infty\right\} \in \{0\,,1\},$$

by Kolmogorov's 0-1 law. □

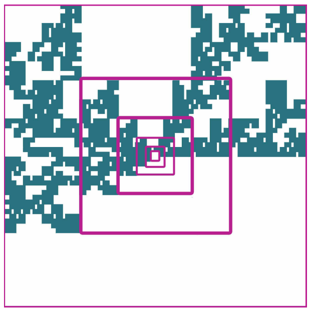

Figure 2: A simulation of a 2-dimensional percolation cluster [the shaded region] for $\Pi_{0.8} \cap \mathcal{V}_5$. The nested squares delineate the cubical shells $\mathcal{S}_0$ through $\mathcal{S}_5$, from smallest to largest; all but $\mathcal{S}_0$ are cubical annuli. The various parts of the shaded areas are independent from shell to shell. Percolation in different cubical annuli results from independent branching processes.

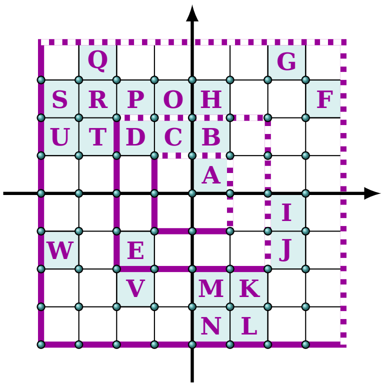

Figure 3: *A 2-dimensional fractal percolation schematic.* Each square is indexed by its southwest corner. Percolation is independent between the thickset cubical annuli, and is the result of a coupled branching process in each cube $\mathcal{V}_k$; i.e., the random sets $\{A\}$, $\{B, \ldots, E\}$ and $\{F, G, \ldots, W\}$ are the result of running independent branching processes. *The enumeration scheme.* Every annulus is enumerated by its index; this indexing enduces a partial ordering of cubes in each shell. Thus, cubes in $\mathcal{S}_0$ are enumerated before those in $\mathcal{S}_1$ and so on. Percolation in the shell $\mathcal{S}_k$ is the result of percolation in $\mathcal{V}_k$ according to the description that follows eq. (3.2). Each $\mathcal{V}_k$ is subdivided into four cubes [the four quadrants] which we enumerate lexicographically; each quadrant is canonically associated with a vector in $\{-1\,,1\}^2$, depending on the sign of the various coordinates, and by lexicographic order of the quadrants we mean the order of these vectors. When $d = 2$, as is the case in the above schematic, we have $1\text{st} > 4\text{th} > 2\text{nd} > 3\text{rd}$ since $(1,1) > (1,-1) > (-1,1) > (-1,-1)$. Each of those cubes is then divided into four cubes; those are again enumerated lexicographically, etc. This enumeration procedure is continued until we are left with cubes of size 1 which are now—for the purposes of the figures—enumerated lexicographically. The resulting scheme for enumerating size-one cubes yields an isomorphism between the percolation set and a certain random forest. That scheme is illustrated further in Figure 5.

A word of notational caution at this point: We will use $F \sim_{(\mathrm{R})} \Pi_p$ as shorthand for $\mathrm{P}\{F \sim_{(\mathrm{R})} \Pi_p\} = 1$. However, when we condition on a given configuration from this a.s. event, $F \sim_{(\mathrm{R})} \Pi_p$ reverts to its original meaning.

As we have noted already, the [microscopic] fractal percolation set $\Pi_{p,\infty}(\mathcal{V}_k)$ is nonvoid if and only if $p > 2^{-d}$. The large-scale analogue of becoming nonvoid is to become unbounded. The following shows that the large-scale result takes a different form than its small-scale counterpart in the critical case $p = 2^{-d}$.

Lemma 3.3. *Π_p is almost surely unbounded if $p \geqslant 2^{-d}$ and it is almost surely bounded if $p < 2^{-d}$.*

We will be interested only in Π_p when it is unbounded; the preceding tells us that we want to consider only values of $p \in [2^{-d}\,, 1]$. The said condition on p will appear several times in the sequel for this very reason.

Proof. By the Borel–Cantelli lemma for independent events, the random set Π_p is unbounded a.s. if $\sum_{k=0}^{\infty} \mathrm{P}\{\Pi_p \cap \mathcal{S}_k \neq \varnothing\} = \infty$; otherwise, Π_p is bounded a.s.

Since $\Pi_p \cap \mathcal{S}_k \subseteq \Pi_p \cap \mathcal{V}_k$, the probability that $\Pi_p \cap \mathcal{S}_k$ is nonempty is at most the probability that a Galton–Watson branching process with mean branch rate $2^d p$ survives after $k+1$ generations. We shall denote by Z_k the number of descendants in the k-th generation. When $p < 2^{-d}$, the said Galton–Watson process is strictly subcritical. It is well-known that $\mathrm{E}(Z_k) = (\mathrm{E}Z_1)^k = (2^d p)^k$. Since $p < 2^{-d}$, the simple bound $\mathrm{P}\{Z_k \geqslant 1\} \leqslant \mathrm{E}(Z_k)$ ensures that

$$\sum_{k=1}^{\infty} \mathrm{P}\{\Pi_p \cap \mathcal{S}_k \neq \varnothing\} \leqslant \sum_{k=1}^{\infty} \mathrm{P}\{\Pi_p \cap \mathcal{V}_k \neq \varnothing\} \leqslant \sum_{k=1}^{\infty} \mathrm{E}(Z_{k+1}) < \infty.$$

Thus, we conclude that Π_p is a.s. bounded when $p < 2^{-d}$.

By the monotonicity of $p \mapsto \Pi_p$, it remains to prove that $\Pi_{2^{-d}}$ is a.s. unbounded. One can define fractal percolation on any dyadic cube $Q \in \mathsf{D}$ in much the same way as we defined it on $\mathcal{V}_k$. Next we note that if $k \geqslant 2$, then we obtain the following by first selecting a cube $Q \in \mathsf{D}_\mathsf{k}$ in $\mathcal{V}_k$ and then another cube in $\mathsf{D}_{\mathsf{k}-1}$ in $\mathcal{S}_k$:

$$\mathrm{P}\left\{\Pi_{2^{-d}} \cap \mathcal{S}_k \neq \varnothing\right\} \geqslant p^2 \mathrm{P}(E_k); \tag{3.3}$$

where E_k denotes the event that fractal percolation with $p = 2^{-d}$ on a dyadic cube of side 2^{k-1} does not become void in $k-1$ steps. In other words, up to a constant multiplicative factor which does not depend on k, the probability $\mathrm{P}\{\Pi_{2^{-d}} \cap \mathcal{S}_k \neq \varnothing\}$ is bounded below by the probability that a certain critical Galton–Watson does not become extinct in its first k generations. A well-known theorem of Kolmogorov [14] asserts that $\lim_{k\to\infty} k\mathrm{P}\{Z_k > 0\} = 2\sigma^{-2}$ for that critical Galton–Watson branching process, where σ^2 denotes the variance of the offspring distribution; see also Kesten et al [8] and Lyons et al [16]. This fact implies that for every $\epsilon > 0$ there exists a positive integer k_0 such that if $k > k_0$ then $\mathrm{P}\{Z_k > 0\} \geqslant c/k$, where $c = 2\sigma^{-2} - \epsilon$. We may select $0 < \epsilon < 2\sigma^{-2}$ and use (3.3) in order to deduce that

$$\sum_{k=0}^{\infty} \mathrm{P}\{\Pi_{2^{-d}} \cap \mathcal{S}_k \neq \varnothing\} \geqslant p^2 \sum_{k=k_0}^{\infty} \mathrm{P}\{Z_k \geqslant 1\} \geqslant p^2 c \sum_{k=k_0}^{\infty} k^{-1} = \infty,$$

Thus, there exists $k_0 > 1$ such that

$$\sum_{k=0}^{\infty} \mathrm{P}\{\Pi_{2^{-d}} \cap \mathcal{S}_k \neq \varnothing\} \geqslant p^2 \sum_{k=k_0}^{\infty} \mathrm{P}\{Z_k \geqslant 1\} \geqslant p^2 k_0^{-1} \sum_{k=k_0}^{\infty} k^{-1} = \infty,$$

which concludes the proof. □

The following is the result of an elementary computation.

Lemma 3.4. *If $x \in \mathcal{S}_k$ for some $k \geqslant 1$, then* $\mathrm{P}\{x \in \Pi_p\} = p^{k+1}$. *If $x, y \in \mathcal{S}_k$ then* $\mathrm{P}\{x, y \in \Pi_p\} = p^{2k+2-\lambda(x,y)}$*, where*

$$\lambda(x\,,y) := (k+1) - \min\left\{n \geqslant 0 :\ \exists Q \in \mathsf{D}_n \text{ such that } x, y \in Q\right\}, \tag{3.4}$$

and $\min\varnothing := k+1$.

According to (3.4), $0 \leqslant \lambda(x\,,y) \leqslant k+1$ for all $x, y \in \mathcal{S}_k$. Both bounds can be achieved: $\lambda(x\,,y) = 0$ when the most recent common ancestor of x and y in the branching process is the root $\mathcal{V}_k$; and $\lambda(x\,,y) = k+1$ when $y = x$.

We will use the preceding in order to prove the following.

Theorem 3.5. *For all non random sets $F \subset \mathbb{R}^d$,*

$$\mathrm{Dim}_{\mathrm{H}}(F) = -\log_2 \inf\left\{p \in [2^{-d}, 1] :\ F \sim_{(\mathrm{R})} \Pi_p\right\},$$

where $\log_2$ denotes the usual base-2 logarithm.

Remark 3.6. Theorem 3.5 can be recast as follows: If $\mathrm{Dim}_{\mathrm{H}}(F) > -\log_2 p$, then $F \sim_{(\mathrm{R})} \Pi_p$; otherwise if $\mathrm{Dim}_{\mathrm{H}}(F) < -\log_2 p$, then $F \not\sim_{(\mathrm{R})} \Pi_p$. The case that $\mathrm{Dim}_{\mathrm{H}}(F) = -\log_2 p$ is not decided by a dimension criterion. That case can be decided by a more delicate capacity criterion. We will not pursue those refinements since we will not need them.

Proof. For every set $F \subset \mathbb{R}^d$ define an integer set $F^z = \phi(F)$ to be the "pixelization" of F. To be concrete, if $x \in \mathbb{R}^d$ falls in (the "semi-open") 1×1 cube Q, then let $\phi(x)$ denotes the south-west corner of Q. This procedure defines F^z canonically now.

Observe that $F \sim_{(\mathrm{R})} \Pi_p$ iff $F^z \sim_{(\mathrm{R})} \Pi_p$, and $\mathrm{Dim}_{\mathrm{H}}(F) = \mathrm{Dim}_{\mathrm{H}}(F^z)$; see Lemma 6.1 of Barlow and Taylor [2]. Therefore, we may assume without loss of generality that F is a subset of $\mathbb{Z}^d$; otherwise, we can replace F by F^z everywhere throughout the remainder of the proof. From now on we consider only $F \subseteq \mathbb{Z}^d$.

Let $k \geqslant 0$ be an arbitrary integer. We can find dyadic cubes $Q_1, \ldots, Q_m \subset \mathcal{S}_k$ such that:

1. For every $1 \leqslant i \leqslant m$, the sidelength of Q_i is $2^{\ell_i} \geqslant 1$ for some integer $0 \leqslant \ell_i \leqslant k+1$;
2. $(F \cap \mathcal{S}_k) \subseteq \cup_{i=1}^m Q_i$; and
3. $\mathcal{N}_{\log_2(1/p)}(F\,, \mathcal{S}_k) = \sum_{i=1}^m p^{k-\ell_i+1}$; see Remark 2.1 for justification.

Thus, we obtain

$$\mathrm{P}\{\Pi_p \cap Q_i \neq \varnothing\} = \mathrm{P}\{\Pi_{p,k}(\mathcal{V}_k) \cap Q_i \neq \varnothing\} \leqslant \mathrm{P}\{\Pi_{p,\ell_i-1}(\mathcal{V}_k) \cap Q_i \neq \varnothing\} = p^{k+1-\ell_i}.$$

For the inequality we have used the fact that $\Pi_{p,i}(\mathcal{V}_k) \supset \Pi_{p,i+1}(\mathcal{V}_k)$ for every integer $i \geqslant -1$. It follows that

$$\mathrm{P}\{F \cap \Pi_p \cap \mathcal{S}_k \neq \varnothing\} \leqslant \sum_{i=1}^m \mathrm{P}\{\Pi_p \cap Q_i \neq \varnothing\} \leqslant \mathcal{N}_{\log_2(1/p)}(F\,, \mathcal{S}_k). \tag{3.5}$$

The preceding holds for all $p \in (0\,,1]$. Suppose for the moment that $\log_2(1/p) > \mathrm{Dim}_{\mathrm{H}}(F)$. Then, $\sum_{k=0}^\infty \mathcal{N}_{\log_2(1/p)}(F\,, \mathcal{S}_k) < \infty$ by the definition of $\mathrm{Dim}_{\mathrm{H}}$; and (3.5) implies that

$$\sum_{k=0}^\infty \mathrm{P}\{F \cap \Pi_p \cap \mathcal{S}_k \neq \varnothing\} < \infty.$$

The non recurrence of F for Π_p follows by the Borel–Cantelli lemma.

We have proved that if $p \in (0\,, 2^{-\mathrm{Dim}_{\mathrm{H}}(F)})$, then F is not recurrent for Π_p. It now remains to show that

$$\text{If } \mathrm{Dim}_{\mathrm{H}}(F) > \delta > 0 \text{ and } p \in (2^{-\delta}\,, 1), \text{ then } F \sim_{(\mathrm{R})} \Pi_p. \tag{3.6}$$

From now on we choose and fix an arbitrary $\delta \in (0\,, \mathrm{Dim}_{\mathrm{H}}(F))$. Note that

$$\sum_{k=0}^{\infty} \mathcal{N}_\delta(F\,, \mathcal{S}_k) = \infty, \tag{3.7}$$

by the definition of $\mathrm{Dim}_{\mathrm{H}}$.

Next we briefly state Theorem 4.2 of Barlow and Taylor [2], in our notation, and adapt it the statements to our settings. This is the version we use in the sequence.

Theorem 3.7 (Barlow and Taylor [2, Theorem 4.2(i)])**.** *For every $A \subset \mathcal{V}_k$ there exists a measure μ, supported only on A, that satisfies*

$$\mu(A) \geqslant \mathcal{N}_\delta(A\,, \mathcal{V}_k). \qquad \textit{Moreover,} \quad \mu(Q) \leqslant c2^{\delta(\ell-k-1)}, \tag{3.8}$$

for every dyadic cube $Q \subseteq \mathcal{V}_k$ with sidelength $2^\ell \geqslant 1$.

We use the above theorem in the following way: Pick an arbitrary $F \subset \mathbb{Z}^d$ and define $F_k = F \cap \mathcal{S}_k \subset \mathcal{V}_k \cap \mathcal{V}_{k-1}^c$. Apply the theorem to every F_k in order to construct a (discrete) measure μ_k, for every k, that is fully supported on F_k and satisfies property (3.8). Since the μ_k's are supported on disjoint sets, they can be pasted together to collectively define a sigma-finite measure $\bar{\mu}$ on $F \subset \mathbb{Z}^d$. It follows readily that there exists a real number $c > 0$, that depends only on the ambient dimension d, such that

$$\bar{\mu}(F \cap \mathcal{S}_k) \geqslant \mathcal{N}_\delta(F\,, \mathcal{S}_k) \qquad \text{and} \qquad \bar{\mu}(Q) \leqslant c2^{\delta(\ell-k-1)},$$

for all integers $k \geqslant 0$ and all dyadic cubes $Q \subset \mathcal{V}_k$ with sidelength $2^\ell \geqslant 1$. The first inequality follows from (3.8) and Remark 2.2 (the measure $\bar{\mu} = c_d^\delta \mu$ where μ is the measure in Theorem 4.2.[2]).

Define a measure μ by normalizing $\bar{\mu}$ on $\mathcal{S}_k$ as follows:

$$\mu(\cdot) := \frac{\bar{\mu}(\cdot)}{\bar{\mu}(\mathcal{S}_k)},$$

in order to conclude the following for all integers $k \geqslant 0$:

(i) $\mu(\mathcal{S}_k) = \mu(F \cap \mathcal{S}_k) = 1$;

(ii) $\mu(Q) \leqslant c2^{\delta(\ell-k-1)}/\mathcal{N}_\delta(F\,, \mathcal{S}_k)$, uniformly for all dyadic cubes $Q \subset \mathcal{S}_k$ with sidelength $2^\ell \geqslant 1$. [This is valid even when $\mathcal{N}_\delta(F\,, \mathcal{S}_k) = 0$, provided that we define $1/0 := \infty$.]

Define

$$\zeta_k := p^{-k-1}\mu(\mathcal{S}_k \cap \Pi_p) = \sum_{x \in \mathcal{S}_k} \frac{\mathbb{1}_{\{x \in \Pi_p\}}}{p^{k+1}} \mu(x),$$

where $\mu(x) := \mu(\{x\})$. Because μ is supported only on $F \cap S_k$, it follows that $\zeta_k > 0$ if and only if $F \cap S_k \cap \Pi_p \neq \varnothing$; that is, as events,

$$\{\omega \in \Omega:\ \zeta_k(\omega) > 0\} = \{\omega \in \Omega:\ F \cap S_k \cap \Pi_p(\omega) \neq \varnothing\}. \tag{3.9}$$

By Lemma 3.4: (i) $\mathrm{E}[\zeta_k] = \mu(\mathcal{S}_k) = 1$; and (ii)

$$\begin{aligned}
\mathrm{E}[\zeta_k^2] &= \sum_{x,y \in \mathcal{S}_k} p^{-\lambda(x,y)}\,\mu(x)\,\mu(y) \\
&= \sum_{j=0}^{k+1} p^{-j}\,(\mu \times \mu)\left\{(x\,,y) \in \mathcal{S}_k^2:\ \lambda(x\,,y) = j\right\} \\
&\leqslant \sum_{j=0}^{k+1} p^{-j}\,(\mu \times \mu)\left\{(x\,,y) \in \mathcal{S}_k^2:\ \lambda(x\,,y) \geqslant j\right\}.
\end{aligned}$$

For a given x, define $C_j(x) := \{y \in \mathcal{S}_k : \lambda(x\,,y) \geqslant j\}$. Because $\mu(\mathcal{S}_k) = 1$,

$$(\mu \times \mu)\{(x\,,y) \in \mathcal{S}_k^2 : \lambda(x\,,y) \geqslant j\} \leqslant \max_{x \in \mathcal{S}_k} \mu(C_j(x)).$$

Therefore, there exists a dyadic cube $Q \in \mathsf{D}_{k+1-j}$ such that $C_j(x) \subset Q$, whence by property (ii) of the measure μ,

$$\mu(C_j(x)) \leqslant \mu(Q) \leqslant \frac{c}{2^{\delta j} \mathcal{N}_\delta(F\,, \mathcal{S}_k)}.$$

This, in turn, implies that

$$\mathrm{E}[\zeta_k^2] \leqslant \frac{c}{\mathcal{N}_\delta(F\,, \mathcal{S}_k)} \cdot \sum_{j=0}^{k} \left(2^\delta p\right)^{-j} \leqslant \frac{c\left[1 - (2^\delta p)^{-1}\right]^{-1}}{\mathcal{N}_\delta(F\,, \mathcal{S}_k)}.$$

For the last inequality we have used the hypothesis of (3.6). The Paley-Zygmund inequality yields $\mathrm{P}\{\zeta_k > 0\} \geqslant (\mathrm{E}[\zeta_k])^2/\mathrm{E}[\zeta_k^2]$. Therefore, our bounds for the two first moments of ζ_k and equation (3.9) together lead us to

$$\mathrm{P}\{\Pi_p \cap F \cap \mathcal{S}_k \neq \varnothing\} \geqslant \mathcal{N}_\delta(F\,, \mathcal{S}_k) \cdot \frac{\left[1 - (2^\delta p)^{-1}\right]}{c}.$$

Thus, it follows from(3.7) that $\sum_{k=0}^\infty \mathrm{P}\{\Pi_p \cap F \cap \mathcal{S}_k \neq \varnothing\} = \infty$. The independence half of the Borel–Cantelli lemma implies (3.6). □

Let us close this section with a quick application of Theorem 3.5.

Corollary 3.8. *For each* $p \in (0\,,1]$, $\mathrm{Dim}_{\mathrm{H}}(\Pi_p) = (d + \log_2 p)^+$ *a.s.*

This result has content only when $p \in [2^{-d}, 1]$; see Lemma 3.3. In particular, it states that the dimension of fractal percolation is 0 at criticality.

Proof. The proof uses the replica argument of Peres [20] without need for essential changes. More specifically, let Π'_q denote a fractal percolation set with parameter $q \in (0\,,1]$ such that Π_p and Π'_q are independent. Because $\Pi_p \cap \Pi'_q$ has the same distribution as Π_{pq}, it follows that

$$\mathrm{P}\{\Pi_p \sim_{(\mathrm{R})} \Pi'_q\} = \mathrm{P}\{\Pi_{pq} \text{ is unbounded}\}.$$

Thus, by Lemma 3.3, we have

$$\mathrm{P}\{\Pi_p \sim_{(\mathrm{R})} \Pi'_q\} = \begin{cases} 1 & \text{if } p \geqslant q^{-1}2^{-d}, \\ 0 & \text{if } p < q^{-1}2^{-d}. \end{cases}$$

We may first condition on Π_p and then appeal to Theorem 3.5 in order to see that $\mathrm{Dim}_{\mathrm{H}}(\Pi_p) = -\log_2(q_c)$, where q_c is the critical constant $q \in (0\,,1]$ such that $pq2^d \geqslant 1$. Because $q_c = p^{-1}2^{-d}$ the corollary follows. □

Remark 3.9. By Theorem 3.5—see also Remark 3.6—and thanks to Corollary 3.8, we can conclude that if $\mathrm{Dim}_{\mathrm{H}}(\Pi_p) + \mathrm{Dim}_{\mathrm{H}}(F) > d$, then $F \sim_{(\mathrm{R})} \Pi_p$; otherwise if $\mathrm{Dim}_{\mathrm{H}}(\Pi_p) + \mathrm{Dim}_{\mathrm{H}}(F) < d$, then $F \not\sim_{(\mathrm{R})} \Pi_p$.

We can now deduce Barlow and Taylor's dimension theorem [Proposition 1.1] from the previous results of this paper. The following is a standard codimension argument; see Taylor [22, Theorem 4].

Proof of Proposition 1.1. Barlow and Taylor [2, Cor. 8.4] have observed that, under the conditions of Proposition 1.1,

$$P^a\{\mathcal{R}_X \sim_{(\mathrm{R})} F\} = 1 \text{ if } \mathrm{Dim}_{\mathrm{H}}(F) > d-\alpha, \text{ and } P^a\{\mathcal{R}_X \not\sim_{(\mathrm{R})} F\} = 1 \text{ if } \mathrm{Dim}_{\mathrm{H}}(F) < d-\alpha.$$

The preceding is an immediate consequence of Proposition 8.2 of [2], which is a variation of Lamperti's test [15]; the general form of this sort of Lamperti's test is in fact Corollary 2.4, which we will prove in due time.

We apply the preceding observation conditionally, with $F := \Pi_p$, where $p \in (2^{-d}, 1]$ is a fixed parameter. By Corollary 3.8, $\mathrm{Dim}_{\mathrm{H}}(\Pi_p) = d + \log_2 p$ a.s.. Therefore,

$$P^a\{\mathcal{R}_X \sim_{(\mathrm{R})} \Pi_p\} = 1 \text{ if } p > 2^{-\alpha}, \text{ and } P^a\{\mathcal{R}_X \not\sim_{(\mathrm{R})} \Pi_p\} = 1 \text{ if } p < 2^{-\alpha}.$$

By the Hewitt–Savage 0-1 law, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X)$ is P^a-a.s. a constant. Choose $p > 2^{-\alpha}$ and assume to the contrary that

$$P^a\{\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) < -\log_2 p\} = 1.$$

Restrict the probability space of the random walk to the full-P^a probability event $\{\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) < -\log_2 p\}$ and fix a realization $F = \mathcal{R}_X$. Theorem 3.5 ensures that $\mathrm{P}\{F \not\sim_{(\mathrm{R})} \Pi_p\} = 1$ for almost all realizations of the random walk. This gives the desired contradiction since $P^a\{\mathcal{R}_X \sim_{(\mathrm{R})} \Pi_p\} = 1$ for P-a.e. realization of Π_p, while

$$1 = \int \mathrm{dP} \int \mathrm{d}P^a\, \mathbb{1}\{\mathcal{R}_X \sim_{(\mathrm{R})} \Pi_p\} = \int \mathrm{d}P^a \int \mathrm{dP}\, \mathbb{1}\{\mathcal{R}_X \sim_{(\mathrm{R})} \Pi_p\} = 0.$$

It follows that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) \geqslant -\log_2 p$ a.s. as long as $p > 2^{-\alpha}$. An analogous argument shows that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) \leqslant -\log_2 p$ whenever $p < 2^{-\alpha}$. Therefore, we conclude that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) = -\log_2(p_c)$ P^a-a.s. where $p_c = 2^{-\alpha}$. □

4 A forest representation of $\mathbb{Z}^d$

If $x \in \mathbb{Z}^d \cap \mathcal{V}_k$ for some integer $k \geqslant 0$, then there exists a unique sequence $Q_0(x), Q_1(x), \ldots, Q_{k+1}(x)$ of dyadic sets such that:

1. $Q_0(x) = \mathcal{V}_k$ and $Q_{k+1}(x) = [x_1, x_1+1) \times \cdots \times [x_d, x_d+1)$;
2. $Q_{i+1}(x) \subset Q_i(x)$ for all $i = 0, \ldots, k$; and
3. $Q_i(x) \in \mathsf{D}_{k-i+1}$ for all $i = 0, \ldots, k+1$.

Conversely, if $Q_0, Q_1, \ldots, Q_{k+1}$ denotes a collection of dyadic cubes such that:

1. $Q_0 = \mathcal{V}_k$;
2. $Q_{i+1} \subset Q_i$ for all $i = 0, \ldots, k$; and
3. $Q_i \in \mathsf{D}_{k-i+1}$ for all $i = 0, \ldots, k+1$;

then there exists a unique point $x \in \mathbb{Z}^d \cap \mathcal{V}_k$ defined unambiguously via $Q_{k+1} = [x_1, x_1 + 1) \times \cdots \times [x_d, x_d+1)$ [equivalently, $x_i := \inf_{y \in Q_{k+1}} y_i$ for $1 \leqslant i \leqslant d$] (see Figure 4). Moreover, $Q_i = Q_i(x)$ for all $0 \leqslant i \leqslant k+1$.

The preceding describes a bijection between the points in $\mathbb{Z}^d \cap \mathcal{S}_k$ and a certain collection of $(k+2)$-chains of dyadic cubes. We can now use this bijection in order to build a directed-tree representation of $\mathbb{Z}^d \cap \mathcal{S}_k$: The vertices of the tree are comprised of all dyadic cubes $Q \in \mathsf{D}$ whose sidelength is $\geqslant 1$. For the [directed] edges of our tree, we draw an arrow from a vertex Q to a vertex Q' if and only if there exists an integer $i = 0, \ldots, k$ and a point $x \in \mathbb{Z}^d$ such that $Q = Q_i(x)$ and $Q' = Q_{i+1}(x)$. The resulting graph is denoted by $\mathcal{T}_k$.

It is easy to observe the following properties of $\mathcal{T}_k$:

1. $\mathcal{T}_k$ is a finite rooted tree, the root of $\mathcal{T}_k$ being $\mathcal{V}_k$;
2. Every ray in $\mathcal{T}_k$ has depth $k+1$;
3. There is a canonical bijection from the rays of $\mathcal{T}_k$ to $\mathbb{Z}^d \cap \mathcal{S}_k$.

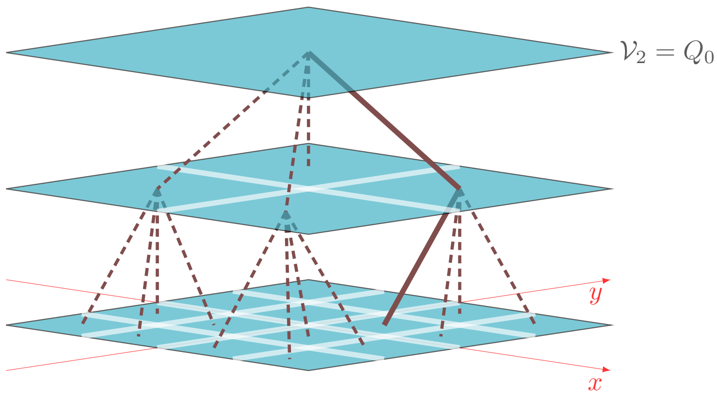

Figure 4: A 2-dimensional tree representation of $\mathcal{V}_2$ with dyadic cubes as nodes. The levels also indicate steps in the percolation branching process. The axes are included in order to help with orientation. Every 1×1 square at the lower level is indexed by its southwest corner. The sequence of cubes in the thickset branch of the tree is $Q_0(0) \supset Q_1(0) \supset Q_2(0)$, in descending order.

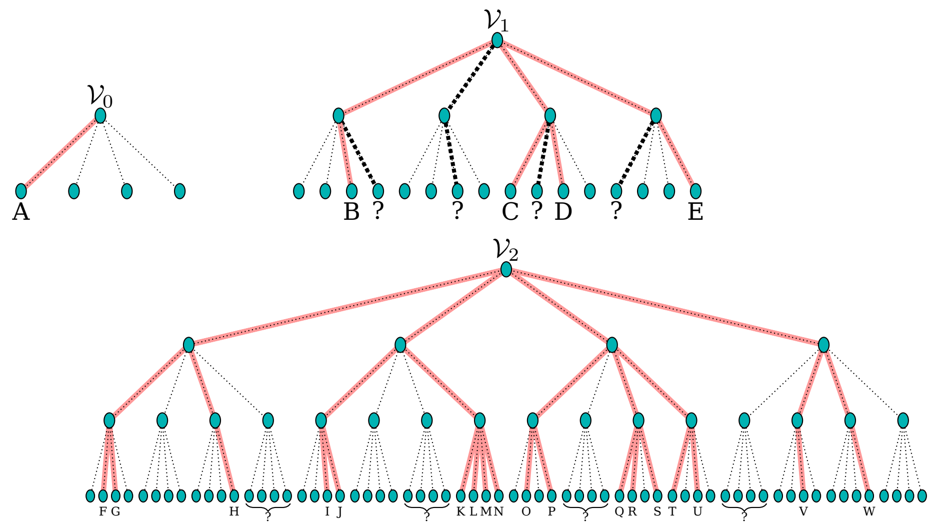

Figure 5: A forest that corresponds to the percolation cluster of Figure 3. The trees correspond to the branching processes in each $\mathcal{V}_k$ as in Figure 4. The thickset purple lines correspond to the surviving population [i.e., the colored squares] in each shell $\mathcal{S}_k$. The question marks signify that we do not have information about that branch of the process if we are allowed to look only at the percolation cluster; they correspond to lattice squares outside the shell $\mathcal{S}_k$.

Since the directed tree $\mathcal{T}_k$ is finite for every $k \geqslant 0$, we can isometrically embed it in $\mathbb{R}^2$ so that the vertices of $\mathcal{T}_k$ that have the maximal depth lie on the real axis.

Of course, there are infinitely-many such possible isometric embeddings; we will choose and fix one [it will not matter which one]. In this way, we can think of every $\mathcal{T}_k$ as a finite rooted tree, drawn in $\mathbb{R}^2$, whose vertices of maximal depth lie on the real axis and whose root lies $k+1$ units above the real axis. Because every $x \in \mathbb{Z}^d \cap \mathcal{S}_k$ has been coded by the rays of $\mathcal{T}_k$, and since those rays can in turn be coded by their last vertex [these are vertices of maximal depth], thus we obtain a bijection π_k that maps each point $x \in \mathbb{Z}^d \cap \mathcal{S}_k$ to a certain point $\pi_k(x)$ on the real axis of $\mathbb{R}^2$. Note that, in this way, $\{\pi_k(x)\}_{x\in\mathbb{Z}^d\cap\mathcal{S}_k}$ can be identified with a finite collection of points on the real line.

The collection $\{\mathcal{T}_k\}_{k=0}^{\infty}$ is a forest representation of $\mathbb{Z}^d$. We use this representation in order to impose an order relation $\prec$ on $\mathbb{Z}^d$ as follows:

1. If $x \in \mathbb{Z}^d \cap \mathcal{S}_k$ and $y \in \mathbb{Z}^d \cap \mathcal{S}_\ell$ for two integers $k, l \geqslant 0$ such that $k < \ell$, then we declare $x \prec y$;
2. If $x, y \in \mathbb{Z}^d \cap \mathcal{S}_k$ for the same integer $k \geqslant 0$, and $\pi_k(x) \leqslant \pi_k(y)$, then we declare $x \prec y$.

It can be checked, using only first principles, that $\prec$ is in fact a bona fide total order on $\mathbb{Z}^d$. We might sometimes also write $y \succ x$ in place of $x \prec y$.

If we identify $x, y \in \mathbb{Z}^d \cap \mathcal{S}_k$ with 2 rays of the tree $\mathcal{T}_k$, viewed as a tree drawn in $\mathbb{R}^d$ as was described earlier, then $x \prec y$ iff the ray for x lies on, or to the left of, the ray for y. And if $x \in \mathcal{S}_k$ and $y \in \mathcal{S}_\ell$ for $k < \ell$, then our definition of $x \prec y$ stems from the fact that we would like to draw the tree $\mathcal{T}_k$ to the left of the tree $\mathcal{T}_\ell$, as we embed the forest $\mathcal{T}_0, \mathcal{T}_1, \ldots$, tree by tree, isometrically in $\mathbb{R}^2$.

5 Martin capacity of fractal percolation

Now we return to Markov chains. Throughout this section, let $X := \{X_n\}_{n=0}^{\infty}$ denote a transient Markov chain on $\mathbb{Z}^d$. This chain is constructed in the usual way: We have a probability space $(A\,, \mathcal{A}\,, P)$ together with a family $\{P^a\}_{a\in\mathbb{Z}^d}$ of probability measures such that under P^a, the Markov chain begins at $X_0 = a$ for every $a \in \mathbb{Z}^d$. By E^a we mean the corresponding expectation operator for P^a for all $a \in \mathbb{Z}^d$ [$E^a(f) := \int f\,\mathrm{d}P^a$].

We assume that the Markov chain is independent of the fractal percolations. The two processes are jointly constructed on $(A \times \Omega\,, \mathcal{A} \times \mathcal{F}, P \times \mathrm{P})$. We write $\mathbb{P}^a := P^a \times \mathrm{P}$ and $\mathbb{E}^a$ the corresponding expectation operator [$\mathbb{E}^a(f) := \int f\,\mathrm{d}\mathbb{P}^a$].

As is well known, the transience of X is equivalent to seemingly-simpler condition that $g(x\,,x) < \infty$ for all $x \in \mathbb{Z}^d$. This is because $g(a\,,x) \leqslant g(x\,,x)$ for all $x \in \mathbb{Z}^d$; in fact, one can apply the strong Markov property to the first hitting time of $x \in \mathbb{Z}^d$ in order to see that

$$g(a\,,x) = g(x\,,x) \cdot P^a\{X_n = x \text{ for some } n \geqslant 0\} \qquad \text{for every } x \in \mathbb{Z}^d. \tag{5.1}$$

We define an equivalence relation on $\mathbb{Z}^d$ as follows: For all $x, y \in \mathbb{Z}^d$ we write "$x \leftrightarrow y$" when there exists an integer $k \geqslant 0$ such that x and y are both in the same shell $\mathcal{S}_k$. Symbol $x \not\leftrightarrow y$ denotes that x and y are in different shells.

If μ is a probability measure on $\mathbb{Z}^d$, then we define two "energy forms" for μ. The first form is defined, for every fixed $a \in \mathbb{Z}^d$, as

$$I(\mu\,; a) := \sum\sum_{\substack{x,y\in\mathbb{Z}^d:\\ x\not\leftrightarrow y}} \frac{g(x\,,y)}{g(a\,,y)} \mu(x)\,\mu(y),$$

where $\mu(z) := \mu(\{z\})$ as before. Recall the definition (3.4) of the pairing $(x\,,y) \mapsto \lambda(x\,,y)$. Our second definition of energy requires an additional parameter $p \in (0\,,1]$, and is defined as follows:

$$I_p(\mu\,; a) := \sum\sum_{\substack{x,y\in\mathbb{Z}^d:\\ x\leftrightarrow y}} \frac{g(x\,,y)}{g(a\,,y)} p^{-\lambda(x,y)} \mu(x)\mu(y).$$

Clearly,

$$I(\mu\,;a) + I_1(\mu\,;a) = \sum\sum_{x,y\in\mathbb{Z}^d} \frac{g(x\,,y)}{g(a\,,y)}\,\mu(x)\,\mu(y)$$

coincides with the *Martin energy of* μ [3].

Finally we define a quantity that can be thought of as a kind of "graded Martin capacity" associated to X: For any set F, $p \leqslant 1$ and $a \in \mathbb{Z}^d$, define the Martin p-capacity by

$$c_p(F;a) := \sup_{\substack{F_0\subseteq F:\\ F_0 \text{ finite}}} \left[\inf_{\mu\in M_1(F_0)} \left\{I(\mu\,;a) + I_p(\mu\,;a)\right\}\right]^{-1}. \tag{5.2}$$

The set function c_1 is the same Martin capacity that appeared earlier in (2.7). It might help to observe that the Martin p-capacity satisfies the following monotonicity property:

$$\text{If } F \subseteq G \quad \text{then} \quad c_p(F;a) \leqslant c_p(G;a). \tag{5.3}$$

The main result of this section can be stated as follows.

Theorem 5.1. *If $F \subseteq \mathbb{Z}^d$ is non-random, then for all $a \in \mathbb{Z}^d$,*

$$\tfrac{1}{2}c_p(F;a) \leqslant \mathbb{P}^a\{X_n \in \Pi_p \cap F \text{ for some } n \geqslant 0\} \leqslant 128c_p(F;a),$$

where c_p is defined by (5.2)*.*

Theorem 5.1 implies the following.

Corollary 5.2. *If $F \subseteq \mathbb{Z}^d$ is recurrent for X P^a-a.s. and $a := X_0 \in \mathbb{Z}^d$ and F are non-random, then*

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) = -\log_2 p_c(F\,;a) \qquad P^a\text{-a.s.},$$

where

$$p_c(F\,;a) := \inf\left\{p \in [2^{-d},1] : \inf_{\substack{G\subset\mathbb{Z}^d:\\ G \text{ is cofinite}}} c_p(F\cap G\,;a) > 0\right\},$$

and $\{c_p(\bullet\,;a)\}_{p\leqslant 1}$ is defined in (5.2) *above.*

Proof. Suppose that there exists a number $p \in [2^{-d}\,,1]$ for which

$$\tau(p) := \tfrac{1}{2}\inf c_p(F\cap G\,;a) > 0,$$

where the infimum is computed over all cofinite sets $G \subset \mathbb{Z}^d$. Define

$$G_N := \left\{x \in \mathbb{Z}^d :\ \|x\| > N\right\} \qquad \text{for all } N \geqslant 1.$$

According to Theorem 5.1,

$$\inf_{N\geqslant 1} \mathbb{P}^a\left\{\mathcal{R}_X \cap \Pi_p \cap F \cap G_N \neq \varnothing\right\} \geqslant \tau(p) > 0.$$

It follows from this and elementary properties of probabilities that

$$\mathbb{P}^a\left\{\mathcal{R}_X \cap \Pi_p \cap F \cap G_N \neq \varnothing \text{ for infinitely many } N \geqslant 1\right\} \geqslant \tau(p) > 0.$$

This in turn shows that

$$\mathbb{P}^a\{\mathcal{R}_X \cap F \sim_{(\mathrm{R})} \Pi_p\} \geqslant \tau(p) > 0.$$

We may apply Lemma 3.2, conditionally on $\mathcal{R}_X$, in order to deduce from the preceding that $\mathcal{R}_X \cap F \sim_{(\mathrm{R})} \Pi_p$ a.s. $[\mathbb{P}^a]$. In particular, we apply Theorem 3.5, once again conditionally on $\mathcal{R}_X$, in order to see that

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) \geqslant -\log_2 p \qquad \mathbb{P}^a\text{-a.s.} \tag{5.4}$$

Optimize over our choice of p to see that

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) \geqslant -\log_2 p_c(F\,;a) \qquad \mathbb{P}^a\text{-a.s.} \tag{5.5}$$

For the other bound, it is enough to consider the case that $p_c \in (2^{-d}\,, 1]$ since $p_c = 2^{-d}$ yields a trivial bound. Thus, we can consider instead a number $p \in [2^{-d}\,, 1)$ such that $\inf c_p(F \cap G\,; a) = 0$, where once again the infimum is over all cofinite sets $G \subset \mathbb{Z}^d$. It is easy to deduce from this choice of p and Theorem 5.1 that

$$\lim_{N\to\infty} \mathbb{P}^a\{\mathcal{R}_X \cap \Pi_p \cap F \cap G_N \neq \varnothing\} = 0.$$

Since the random walk X is transient, and because $\mathcal{R}_X \cap F$ is a.s. recurrent, elementary properties of probabilities imply that $\mathbb{P}^a\{\mathcal{R}_X \cap F \sim_{(\mathrm{R})} \Pi_p\} = 0$. Therefore, we may apply Theorem 3.5, one more time conditionally on $\mathcal{R}_X$, in order to see that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) \leqslant -\log_2 p$. Optimize over our choice of p to see that

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) \leqslant -\log_2 p_c(F\,;a) \qquad \mathbb{P}^a\text{-a.s.}$$

The corollary follows. □

Theorem 5.1 has a number of other consequences as well. The following is a universal estimate on the expected Martin p-capacity of the range of the fractal percolation set in a shell $\mathcal{S}_k$.

Corollary 5.3. *For every point $a \in \mathbb{Z}^d$, integers $k \geqslant 0$, non-random finite sets $F \subset \mathbb{Z}^d$, and percolation parameters $p, q \in [2^{-d}\,, 1]$,*

$$256^{-1} c_{pq}(F; a) \leqslant \mathrm{E}\left[c_p(\Pi_q \cap F; a)\right] \leqslant 256 c_{pq}(F; a).$$

Proof of Corollary 5.3. Let Π'_q denote an independent copy of Π_q and denote the corresponding (independent of $\mathbb{P}^a$) measure by P' with corresponding expectation operator E'. Theorem 5.1 ensures that

$$\mathbb{P}^a\{\mathcal{R}_X \cap \Pi_p \cap \Pi'_q \cap F \neq \varnothing\} \leqslant 128 c_p(\Pi'_q \cap F\,; a) \qquad \mathrm{P}'\text{-a.s.}$$

Therefore we integrate $[\mathrm{P}']$ in order to see that

$$(\mathbb{P}^a \times \mathrm{P}')\{\mathcal{R}_X \cap \Pi_p \cap \Pi'_q \cap F \neq \varnothing\} \leqslant 128\mathrm{E}[c_p(\Pi_q \cap F; a)]. \tag{5.6}$$

For the other bound, we recall that $\Pi_p \cap \Pi'_q$ has the same law $[\mathrm{P} \times \mathrm{P}']$ as Π_{pq} does $[\mathrm{P}]$. Therefore, Theorem 5.1 implies that

$$\begin{aligned}(\mathbb{P}^a \times \mathrm{P}')\{\mathcal{R}_X \cap \Pi_p \cap \Pi'_q \cap F \neq \varnothing\} &= \mathbb{P}^a\{\mathcal{R}_X \cap \Pi_{pq} \cap F \neq \varnothing\} \\ &\geqslant \tfrac{1}{2} c_{pq}(F; a).\end{aligned} \tag{5.7}$$

Together, (5.6) and (5.7) yield $c_{pq}(F; a) \leqslant 256\mathrm{E}[c_p(\Pi_q \cap F; a)]$. The other bound in the statement follows similarly. □

The second consequence of Theorem 5.1 is a Lamperti-type condition on recurrence that was stated in Corollary 2.4.

Proof of Corollary 2.4. Consider the stopping times $\{T_k\}_{k=0}^{\infty}$ defined by

$$T_k := \inf\{n \geqslant 0 :\ X_n \in F \cap \mathcal{S}_k\} \qquad \text{for all } k \geqslant 0,$$

where $\inf \varnothing := \infty$. Theorem 5.1 ensures that

$$P^z\{T_m < \infty\} \asymp c_1(F \cap \mathcal{S}_m; z) \tag{5.8}$$

for all integers m, and $z \in \mathbb{Z}^d$.

Of course, $F \sim_{(\mathrm{R})} \mathcal{R}_X$ if and only if $\sum_{k=0}^{\infty} \mathbb{1}_{\{T_k<\infty\}} = \infty$. Therefore, if $\sum_{k=0}^{\infty} c_1(F \cap \mathcal{S}_k; a) < \infty$, then the Borel–Cantelli lemma and (5.8) together imply that $\mathcal{R}_X \not\sim_{(\mathrm{R})} F$. Note that this portion does not require the Lamperti condition (2.8). The complementary half of the corollary does.

If, on the other hand, $\sum_{k=0}^{\infty} c_1(F \cap \mathcal{S}_k; a) = \infty$, then (5.8) ensures that $\sum_{k=1}^{\infty} P^a\{T_k < \infty\} = \infty$. A standard second moment argument reduces our problem to showing the existence of a positive constant C_0 such that

$$\sum_{k=0}^{N}\sum_{j=k}^{N} P^a\{T_k<\infty, T_j<\infty\} \leqslant C_0 \left(\sum_{k=0}^{N} P^a\{T_k<\infty\}\right)^2, \tag{5.9}$$

as $N \to \infty$. This is what we aim to prove.

By the strong Markov property,

$$\begin{aligned} P^a\{T_k < T_{k+l} < \infty\} &= E^a\left[\mathbb{1}_{\{T_k<\infty\}} P^{X_{T_k}}\{T_{k+l}<\infty\}\right] \\ &\leqslant 128 E^a\left[\mathbb{1}_{\{T_k<\infty\}} c_1(F\cap\mathcal{S}_{k+l}; X_{T_k})\right] \\ &\leqslant 128 P^a\{T_k<\infty\} \sup_{z\in\mathbb{Z}^d\cap\mathcal{S}_k} c_1(F\cap\mathcal{S}_{k+l}; z). \end{aligned}$$

Let us observe that, thanks to (2.8), there exists a finite constant $K > 1$ such that $g(x\,,y) \leqslant Kg(a\,,y)$ whenever $x \in \mathcal{S}_n$ and $y \in \mathcal{S}_m$ for integers $m > n \geqslant K$ such that $m \geqslant n + K$. Thus, it follows readily from the definition (5.2) of c_1 that

$$\max_{z\in\mathcal{S}_k\cap\mathbb{Z}^d} c_1(F\cap\mathcal{S}_{k+l}; z) \leqslant K c_1(F\cap\mathcal{S}_{k+l}; a),$$

uniformly for all integers $k, l \geqslant K$. In accord with (5.8),

$$P^a\{T_k < T_{k+l} < \infty\} \leqslant 256 K P^a\{T_k<\infty\} P^a\{T_{k+l}<\infty\}, \tag{5.10}$$

whenever $k, l \geqslant K$.

Similarly, we can appeal to (2.8) in order to find a finite constant $K' > 1$ such that

$$P^a\{T_{k+l} < T_k < \infty\} \leqslant 256 K' P^a\{T_{k+l}<\infty\} P^a\{T_k<\infty\}, \tag{5.11}$$

as long as $k, l \geqslant K'$. Let $K_0 := 256 \max(K\,, K')$. Because

$$P^a\{T_k<\infty\,, T_{k+l}<\infty\} = P^a\{T_k<T_{k+l}<\infty\} + P^a\{T_{k+l}<T_k<\infty\},$$

Eq. (5.10) and eq. (5.11) together imply that

$$\begin{aligned} &\sum_{k=K_0}^{N}\sum_{j=k}^{N} P^a\{T_k<\infty\,, T_j<\infty\} \\ &\leqslant \sum_{k=K_0}^{N}\sum_{j=k+K_0}^{N} P^a\{T_k<\infty\,, T_j<\infty\} + \sum_{k=K_0}^{N}\sum_{j=k}^{k+K_0-1} P^a\{T_k<\infty\,, T_j<\infty\} \\ &\leqslant K_0\left[\sum_{k=0}^{N} P^a\{T_k<\infty\}\right]^2 + K_0\sum_{k=0}^{N} P^a\{T_k<\infty\}. \end{aligned} \tag{5.12}$$

Since $\sum_{k=0}^{N} P^a\{T_k < \infty\} \to \infty$ as $N \to \infty$, we obtain (5.9) if $C_0 = 2K_0$ for all N large. The corollary follows immediately. $\square$

Let us mention a final corollary of Theorem 5.1. That corollary presents a more tractable formula for $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F)$, valid under the Lamperti-type condition (2.8).

Corollary 5.4. *Let $F \subset \mathbb{Z}^d$ and $X_0 := a \in \mathbb{Z}^d$ be non random. Then,* $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) \leqslant -\log_2 p_c(F\,;a)$ *a.s.* $[P^a]$*, where*

$$\begin{aligned} p_c(F\,;a) :&= \sup\Big\{p \in [2^{-d}\,,1]:\ \sum_{k=0}^{\infty} c_p(F \cap \mathcal{S}_k; a) < \infty\Big\} \\ &= \inf\Big\{p \in [2^{-d}\,,1]:\ \sum_{k=0}^{\infty} c_p(F \cap \mathcal{S}_k; a) = \infty\Big\}. \end{aligned}$$

If, in addition, (2.8) *holds, then*

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) = -\log_2 p_c(F\,;a) \qquad \textit{a.s. } [\mathbb{P}^a].$$

Remark 5.5. If X is a random walk that satisfies the conditions of Proposition 1.1 and starts at 0, then our previous comments in Example 2.5 imply that

$$c_p(F \cap \mathcal{S}_k\,;0) \asymp 2^{k\alpha} \sum\sum_{x,y \in \mathcal{S}_k} g(x\,,y) p^{-\lambda(x,y)}\, \mu(x)\, \mu(y).$$

Proof of Corollary 5.4. First we prove the upper bound on $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F)$.

If $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) > -\log_2 p$ for some $p \in (2^{-d}\,,1]$, then Theorem 3.5 ensures that $X \sim_{(\mathrm{R})} \Pi_p$; see especially Remark 3.6. This, the easy half of the Borel–Cantelli lemma, and Theorem 5.1 together imply that $\sum_{k=0}^{\infty} c_p(F \cap \mathcal{S}_k; a) = \infty$. Optimize over $p \in (2^{-d}\,,1]$ in order to deduce that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) \leqslant -\log_2 p_c(F\,;a)$ $\mathbb{P}^a$-a.s. In the reverse direction we assume that (2.8) holds, and strive to show that

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F) \geqslant -\log_2 p_c(F\,;a) \qquad \mathbb{P}^a\text{-a.s.} \tag{5.13}$$

There is nothing to prove if $p_c(F\,;a) = 1$. Therefore, we assume without loss of generality that

$$2^{-d} \leqslant p_c(F\,;a) < 1.$$

According to Theorem 5.1, and thanks to the definition of the critical probability $p_c(F\,;a)$, $\sum_{k=0}^{\infty} \mathbb{P}^a\{\mathcal{R}_X \cap F \cap \Pi_p \cap \mathcal{S}_k \neq \varnothing\} = \infty$ for every $p \in (p_c(F\,;a)\,,1]$. That is,

$$\lim_{N\to\infty} \mathbb{E}^a \tau_N = \infty, \text{ where } \tau_N := \sum_{k=0}^{N} \mathbb{1}\{\mathcal{R}_X \cap F \cap \Pi_p \cap \mathcal{S}_k \neq \varnothing\}. \tag{5.14}$$

Next we verify that there exists a uniform positive constant A so that

$$\mathbb{E}^a\left[\tau_N^2\right] \leqslant A\left(\mathbb{E}^a \tau_N\right)^2 \qquad \text{as } N \to \infty. \tag{5.15}$$

By the Markov property of X and the particular construction of Π_p,

$$\begin{aligned} \mathbb{E}^a\left(\tau_N^2\right) &\leqslant 2 \sum\sum_{0\leqslant j \leqslant k \leqslant N} \mathbb{P}^a\left\{\mathcal{R}_X \cap F \cap \Pi_p \cap \mathcal{S}_j \neq \varnothing\,, \mathcal{R}_X \cap F \cap \Pi_p \cap \mathcal{S}_k \neq \varnothing\right\} \\ &\leqslant 2 \sum\sum_{0\leqslant j \leqslant k \leqslant N} \mathbb{P}^a\left\{\mathcal{R}_X \cap F \cap \Pi_p \cap \mathcal{S}_j \neq \varnothing\right\} \max_{z \in \mathbb{Z}^d \cap \mathcal{S}_j} \mathbb{P}^z\left\{\mathcal{R}_X \cap F \cap \Pi_p \cap \mathcal{S}_k \neq \varnothing\right\}. \end{aligned}$$

Theorem 5.1 then implies that

$$\mathbb{E}^a\left(\tau_N^2\right) \leqslant C \sum\sum_{0\leqslant j \leqslant k \leqslant N} c_p(F \cap \mathcal{S}_j; a) \max_{z \in \mathbb{Z}^d \cap \mathcal{S}_j} c_p(F \cap \mathcal{S}_k; z),$$

where $C := 32768$. We now apply an argument very similar to the one used to produce (5.9) in order to see that there exists an integer $K_* > 1$ such that

$$\max_{z \in \mathbb{Z}^d \cap \mathcal{S}_j} c_p(F \cap \mathcal{S}_k; z) \leqslant K_* c_p(F \cap \mathcal{S}_k; a),$$

as long as $k \geqslant j + K_*$. In this way we find that

$$\mathbb{E}^a\left(\tau_N^2\right) \leqslant CK_* \left(\sum_{j=0}^N c_p(F\cap\mathcal{S}_k\,; a)\right)^2 + \sum\sum_{\substack{0\leqslant j\leqslant k\leqslant N:\\ k<j+K_*}} c_p(F\cap\mathcal{S}_j\,; a) \max_{z\in\mathbb{Z}^d\cap\mathcal{S}_j} c_p(F\cap\mathcal{S}_k\,; z).$$

Since $\sup_{z\in\mathbb{Z}^d} c_p(F\cap\mathcal{S}_k\,; z) \leqslant 2$ [see Theorem 5.1], it follows that

$$\mathbb{E}^a\left(\tau_N^2\right) \leqslant CK_* \left(\sum_{j=0}^N c_p(F\cap\mathcal{S}_k\,; a)\right)^2 + 2K_*\sum_{j=0}^N c_p(F\cap\mathcal{S}_j\,; a).$$

Therefore, Theorem 5.1 shows that $\mathbb{E}^a(\tau_N^2) \leqslant 4CK_*[\mathbb{E}^a\tau_N]^2 + 2K_*\mathbb{E}^a\tau_N$. Because of the 0-1 law [see Lemma 3.2], this and (5.14) together imply that $\tau_N \to \infty$ a.s. $[\mathbb{P}^a]$ as $N \to \infty$. This is another way to state that $\mathcal{R}_X \cap F \sim_{(\mathrm{R})} \Pi_p$ a.s. $[\mathbb{P}^a]$. Theorem 3.5—see, in particular, Remark 3.6—then implies that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X\cap F) \geqslant -\log_2 p$ $\mathbb{P}^a$-a.s. Since $p \in (p_c(F\,;a)\,,1]$ were arbitrary, the lower bound (5.13) follows. □

We conclude this section with a proof of Theorem 5.1.

Proof of Theorem 5.1. Because $\mathbb{P}^a\{X_n \in \Pi_p \cap F \text{ for some } n \geqslant 0\}$ is equal to

$$\sup_{\substack{F_0\subseteq F:\\ F_0 \text{ is finite}}} \mathbb{P}^a\{X_n \in \Pi_p\cap F_0 \text{ for some } n\geqslant 0\},$$

we can assume without loss of generality that F is a finite set.

The first inequality of the proposition follows readily by adapting the second-moment argument of Benjamini et al [3]. The few details follow.

For every $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, there exists a unique positive integer k such that $x \in \mathcal{S}_k$. Let $\Delta(x) := k + 1$ for this pairing of $x \in \mathbb{Z}^d$ and $k \geqslant 1$. Then we define, for all probability measures $\mu \in M_1(F)$, a nonnegative random variable

$$J_\mu := \sum_{n=0}^\infty \frac{\mu(X_n)}{g(a\,,X_n)}\frac{\mathbb{1}_{\{X_n\in\Pi_p\}}}{p^{\Delta(X_n)}},$$

where $\mu(w) := \mu(\{w\})$ for every $w \in \mathbb{Z}^d$. The preceding display contains an almost surely well-defined sum because the summands are non negative and $\mu(X_n)/g(a\,,X_n) \leqslant 1$ a.s. $[P^a]$ for all $n \geqslant 0$. We can therefore rearrange the sum and write

$$J_\mu = \sum_{n=0}^\infty\sum_{x\in\mathbb{Z}^d} \frac{\mathbb{1}_{\{x\}}(X_n)}{g(a\,,x)}\frac{\mathbb{1}_{\Pi_p}(x)}{p^{\Delta(x)}}\mu(x). \tag{5.16}$$

Because $\mathrm{P}\{x\in\Pi_p\} = p^{\Delta(x)}$ for all $x \in \mathbb{Z}^d$,

$$\mathbb{E}^a J_\mu = 1.$$

Similarly, we compute

$$\begin{aligned}\mathbb{E}^a(J_\mu^2) &\leqslant 2\sum_{x,y\in\mathbb{Z}^d}\sum\sum_{m\geqslant n\geqslant 0}\frac{P^a\{X_n = x\,,X_m = y\}}{g(a\,,x)g(a\,,y)}\frac{\mathrm{P}\left\{x,y\in\Pi_p\right\}}{p^{\Delta(x)+\Delta(y)}}\mu(x)\mu(y)\\ &= 2I(\mu\,;a) + 2I_p(\mu\,;a).\end{aligned} \tag{5.17}$$

If $J_\mu > 0$ for some $\mu \in M_1(F)$, then certainly $X_n \in \Pi_p \cap F$ for some $n \geqslant 0$. Therefore, the Paley–Zygmund inequality implies that for every $\mu \in M_1(F)$,

$$\begin{aligned}\mathrm{P}^a\{X_n \in \Pi_p \cap F \text{ for some } n \geqslant 0\} \geqslant \mathrm{P}^a\{J_\mu > 0\} &\geqslant \frac{[\mathrm{E}^a J_\mu]^2}{\mathrm{E}^a(J_\mu^2)} \\ &\geqslant \tfrac{1}{2}\left[I(\mu\,;a) + I_p(\mu\,;a)\right]^{-1}.\end{aligned} \tag{5.18}$$

The left-most quantity does not depend on $\mu \in M_1(F)$; therefore, we may optimize the right-most quantity in (5.18) over all probability measures $\mu \in M_1(F)$ in order to see that $\mathrm{P}^a\{X_n \in \Pi_p \cap F \text{ for some } n \geqslant 0\} \geqslant \frac{1}{2}c_p(F;a)$. This is the desired lower bound on the hitting probability of the theorem.

Next we verify the complementary probability, still assuming without loss of generality that F is finite; that is, $F \subseteq \mathcal{V}_k$ for a nonnegative integer k that is still held fixed throughout. Without loss of generality, we may also assume that

$$\mathrm{P}^a\{X_m \in \Pi_p \cap F \text{ for some } m \geqslant 0\} > 0. \tag{5.19}$$

Otherwise, there is nothing to prove.

In order to establish the more interesting second inequality of the theorem we will need to introduce some notation. Let $\mathcal{X}_n$ denote the sigma-algebra generated by $X_0, \ldots, X_n$ for all $n \geqslant 0$.

Recall that, because of our forest representation of $\mathbb{Z}^d$, we identify every point $\rho \in \mathcal{S}_k \cap \mathbb{Z}^d$ with a ray in a finite tree $\mathcal{T}_k$, which was described in §4. Recall also that $\mathcal{T}_k$ has been embedded in $\mathbb{R}^2$ so that its deepest vertices lie on the real axis of $\mathbb{R}^2$. In this way, we can identify every point $\rho \in \mathcal{S}_k \cap \mathbb{Z}^d$ with a point, which we continue to write as ρ, on the real axis.

For every $\rho \in \mathcal{V}_k \cap \mathbb{Z}^d$, let $\mathcal{P}_\rho$ denote the sigma-algebra generated by the fractal-percolation weights $I_p(Q_0(y)), \ldots, I_p(Q_{\Delta(y)+1}(y))$ for all $y \in \mathbb{Z}^d \cap \mathcal{V}_k$ such that $y \prec \rho$. Similarly, let $\mathcal{F}_\rho$ the sigma-algebra generated by all of the fractal-percolation weights $I_p(Q_0(y)), \ldots, I_p(Q_{\Delta(y)+1}(y))$, where $y \in \mathbb{Z}^d \cap \mathcal{V}_k$ satisfies $y \succ \rho$. If we think of ρ as a maximum-depth vertex of $\mathcal{T}_k$ and the latter is embedded in $\mathbb{R}^2$, as was mentioned earlier, then we can think of $\mathcal{P}_\rho$ as the information, on the fractal percolation, on $\mathcal{T}_k$ that lies to the left of ρ [including ρ]; this is the "$\mathcal{P}$ast" of ρ. Similarly, we may think of $\mathcal{F}_\rho$ as the information to the right of ρ; this is the "$\mathcal{F}$uture" of ρ.

Next we define two "2-parameter martingales," Λ and V as follows:

$$\Lambda_{m,\rho} := \mathrm{E}^a[J_\mu \,|\, \mathcal{X}_m \vee \mathcal{F}_\rho]; \qquad V_{m,\rho} := \mathrm{E}^a[J_\mu \,|\, \mathcal{X}_m \vee \mathcal{P}_\rho],$$

for all $m \geqslant 0$ and $\rho \in \mathcal{V}_k \cap \mathbb{Z}^d$. Because our random walk is independent of the fractal percolation, we may write the following after we appeal to independence:

$$\Lambda_{m,\rho} \geqslant \sum_{n=m}^{\infty} \sum_{\substack{x \in \mathbb{Z}^d: \\ x \succ \rho}} \frac{P^a(X_n = x \,|\, \mathcal{X}_m)}{g(a\,,x)} \frac{\mathrm{P}(x \in \Pi_p \,|\, \mathcal{F}_\rho)}{p^{\Delta(x)}} \mu(x) \cdot \mathbb{1}_{\{X_m = \rho \in \Pi_p \cap F\}}.$$

The Markov property implies the a.s.-inequality,

$$\Lambda_{m,\rho} \geqslant \sum_{\substack{x \in \mathbb{Z}^d: \\ x \succ \rho \\ x \leftrightarrow \rho}} \frac{g(X_m\,,x)}{g(a\,,x)} p^{-\lambda(x,X_m)} \mu(x) \cdot \mathbb{1}_{\{X_m = \rho \in \Pi_p \cap F\}} + \sum_{\substack{x \in \mathbb{Z}^d: \\ x \succ \rho \\ x \not\leftrightarrow \rho}} \frac{g(X_m\,,x)}{g(a\,,x)} \mu(x) \cdot \mathbb{1}_{\{X_m = \rho \in \Pi_p \cap F\}}.$$

We stress, once again, that the ratio of the Green's functions are well defined P^a-a.s.

Similarly,

$$V_{m,\rho} \geqslant \sum_{n=m}^{\infty} \sum_{\substack{x\in\mathbb{Z}^d:\\ x\prec\rho}} \frac{P^a(X_n = x \,|\, \mathcal{X}_m)}{g(a\,,x)} \frac{\mathrm{P}(x\in\Pi_p \,|\, \mathcal{P}_\rho)}{p^{\Delta(x)}} \mu(x)\cdot \mathbb{1}_{\{X_m=\rho\in\Pi_p\cap F\}}$$

$$= \sum_{\substack{x\in\mathbb{Z}^d:\\ x\prec\rho\\ x\leftrightarrow\rho}} \frac{g(X_m\,,x)}{g(a\,,x)} p^{-\lambda(x,X_m)}\,\mu(x)\cdot \mathbb{1}_{\{X_m=\rho\in\Pi_p\cap F\}}$$

$$+ \sum_{\substack{x\in\mathbb{Z}^d:\\ x\prec\rho\\ x\not\leftrightarrow\rho}} \frac{g(X_m\,,x)}{g(a\,,x)}\,\mu(x)\cdot \mathbb{1}_{\{X_m=\rho\in\Pi_p\cap F\}}.$$

Therefore, with probability one,

$$\Lambda_{m,\rho} + V_{m,\rho} \geqslant \sum_{\substack{x\in\mathbb{Z}^d:\\ x\leftrightarrow\rho}} \frac{g(X_m\,,x)}{g(a\,,x)} p^{-\lambda(x,X_m)}\mu(x)\cdot \mathbb{1}_{\{X_m=\rho\in\Pi_p\cap F\}}$$

$$+ \sum_{\substack{x\in\mathbb{Z}^d:\\ x\not\leftrightarrow\rho}} \frac{g(X_m\,,x)}{g(a\,,x)}\mu(x)\cdot \mathbb{1}_{\{X_m=\rho\in\Pi_p\cap F\}}.$$

There are only a countable number of such pairs $(m\,,\rho)$. Therefore, the previous lower bound on $\Lambda_{m,\rho}$ holds, off a single null set, simultaneously for all integers $m \geqslant 0$ and integral points $\rho\in\mathcal{V}_k\cap\mathbb{Z}^d$.

In order to simplify the typesetting, let us write

$$\Lambda_* := \sup_{m\geqslant 0,\rho\in\mathcal{V}_k\cap\mathbb{Z}^d} \Lambda_{m,\rho} \qquad \text{and} \qquad V_* := \sup_{m\geqslant 0,\rho\in\mathcal{V}_k\cap\mathbb{Z}^d} V_{m,\rho}.$$

We might note that, with probability one,

$$\begin{aligned}&\Lambda_* + V_*\\ &\geqslant \sum_{\substack{x\in\mathbb{Z}^d:\\ x\leftrightarrow X_m}} \frac{g(X_m\,,x)}{g(a\,,x)} p^{-\lambda(x,X_m)}\mu(x)\cdot \mathbb{1}_{\{X_m\in\Pi_p\cap F\}} + \sum_{\substack{x\in\mathbb{Z}^d:\\ x\not\leftrightarrow X_m}} \frac{g(X_m\,,x)}{g(a\,,x)}\mu(x)\cdot \mathbb{1}_{\{X_m\in\Pi_p\cap F\}},\end{aligned} \tag{5.20}$$

simultaneously for all integers $m\geqslant 0$.

Now we apply an idea that is, in a different form due to Fitzsimmons and Salisbury [6]. Define a $\mathbb{Z}_+\cup\{\infty\}$-valued random variable M by

$$M := \inf\{m\geqslant 0:\ X_m\in\Pi_p\cap F\},$$

where $\inf\varnothing := \infty$. M is a stopping time with respect to the filtration of the random walk, conditionally on the entire history of the fractal percolation, P-a.s. on $\{\Pi_p\cap F\neq\varnothing\}$.

Consider the event,

$$G := \{\omega\in\Omega:\ M(\omega)<\infty\,,\Pi_p(\omega)\cap F\neq\varnothing\}. \tag{5.21}$$

Hypothesis (5.19) is another way to state $\mathbb{P}^a(G) > 0$. Moreover, (5.20) implies the following key a.s. inequality:

$$\Lambda_* + V_* \geqslant \sum_{x\in\mathbb{Z}^d} \frac{g(X_M\,,x)}{g(a\,,x)} \left\{p^{-\lambda(x,X_M)}\mathbb{1}_{\{x\leftrightarrow X_M\}} + \mathbb{1}_{\{x\not\leftrightarrow X_M\}}\right\}\mu(x)\cdot \mathbb{1}_G.$$

The preceding is valid $\mathbb{P}^a$-a.s. for any probability measure μ on F. We apply it using the following particular choice:

$$\mu(x) := \mathbb{P}^a(X_M = x \mid G) \qquad (x \in \mathbb{Z}^d). \tag{5.22}$$

For this particular choice of $\mu \in M_1(F)$ we obtain the following:

$$\begin{aligned}
&\mathbb{E}^a\left(|\Lambda_* + V_*|^2\right) \\
&\geqslant \mathbb{E}^a\left(\left[\sum_{x\in\mathbb{Z}^d} \frac{g(X_M\,,x)}{g(a\,,x)}\left\{p^{-\lambda(x,X_M)}\mathbb{1}_{\{x\leftrightarrow X_M\}} + \mathbb{1}_{\{x\not\leftrightarrow X_M\}}\right\}\mu(x)\right]^2 \,\middle|\, G\right)\cdot \mathbb{P}^a(G) \\
&\geqslant \left[\mathbb{E}^a\left(\sum_{x\in\mathbb{Z}^d} \frac{g(X_M\,,x)}{g(a\,,x)}\left\{p^{-\lambda(x,X_M)}\mathbb{1}_{\{x\leftrightarrow X_M\}} + \mathbb{1}_{\{x\not\leftrightarrow X_M\}}\right\}\mu(x) \,\middle|\, G\right)\right]^2 \cdot \mathbb{P}^a(G) \\
&= [I(\mu\,;a) + I_p(\mu\,;a)]^2 \cdot \mathbb{P}^a(G),
\end{aligned} \tag{5.23}$$

$\mathbb{P}^a$-a.s., thanks to the Cauchy–Schwarz inequality and out special choice of μ in (5.22). [The conditional expectation is well defined since $\mathbb{P}^a(G) > 0$.]

Recall that the forest representation of $\mathbb{Z}^d$ in §4 identifies $\rho \in \mathcal{V}_k \cap \mathbb{Z}^d$ with a certain finite subset of the real line. With this in mind, we see that $\{\Lambda_{m,\rho}\}_{m\geqslant 0,\rho\in\mathcal{V}_k\cap\mathbb{Z}^d}$ is a 2-parameter martingale under the probability measure $\mathbb{P}^a$, in the sense of Cairoli [5], with respect to the 2-parameter filtration

$$\mathcal{G} := \{\mathcal{X}_m \vee \mathcal{F}_\rho\}_{m\geqslant 0,\rho\in\mathcal{V}_k\cap\mathbb{Z}^d}\,. \tag{5.24}$$

Because $\mathcal{X}$ and $\mathcal{F}$ are independent, the 2-parameter filtration $\mathcal{G}$ satisfies the commutation hypothesis (F4) of Cairoli [5]; see Khoshnevisan [9, §3.4, p. 35] for a more modern account. Therefore, Cairoli's maximal inequality for orthomartingales [9, Corollary 3.5.1, p. 37] implies that $\mathbb{E}^a(\Lambda_*^2) \leqslant 16\sup_{m,\rho}\mathbb{E}^a(\Lambda_{m,\rho}^2)$. This and Jensen's inequality together imply that

$$\mathbb{E}^a(\Lambda_*^2) \leqslant 16\mathbb{E}^a(J_\mu^2).$$

Similarly, we can prove that $\mathbb{E}^a(V_*^2) \leqslant 16\mathbb{E}^a(J_\mu^2)$. Therefore, we may combine these remarks with (5.17) in the following manner:

$$\mathbb{E}^a\left(|\Lambda_* + V_*|^2\right) \leqslant 64\mathbb{E}^a(J_\mu^2) \leqslant 128\left\{I(\mu\,;a) + I_p(\mu\,;a)\right\}. \tag{5.25}$$

Because of the above bound and (5.23), and since $\mathbb{P}^a(G) > 0$ [see (5.19)], it follows that $I(\mu\,;a) + I_p(\mu\,;a)$ is strictly positive and finite. Therefore, we may resolve (5.23) using (5.25) in order to obtain the inequality

$$\mathbb{P}^a(G) \leqslant \frac{128}{I(\mu\,;a) + I_p(\mu\,;a)} \leqslant 128c_p(F\,;a).$$

This completes our proof. □

6 Macroscopic Minkowski dimension

Let us recall [1, 2] that the *macroscopic upper Minkowski dimension* of a set $A \subset \mathbb{Z}^d$ is defined as[4]

$$\overline{\mathrm{Dim}}_{\mathrm{M}}(A) := \limsup_{n\to\infty} n^{-1}\log_2(\mathrm{card}\,(A\cap\mathcal{V}_n)),$$

where $\log_2$ is the usual logarithm in base two.

[4]Barlow and Taylor write $\dim_{\mathrm{UM}}$ in place of our $\overline{\mathrm{Dim}}_{\mathrm{M}}$.

In analogy with the usual [small-scale] theory of the dimensions, $\mathrm{Dim}_{\mathrm{H}}(A) \leqslant \overline{\mathrm{Dim}}_{\mathrm{M}}(A)$ for all sets $A \subseteq \mathbb{Z}^d$; see Barlow and Taylor [2]. The Minkowski dimension is perhaps the most commonly used notion of large-scale dimension, in some form or another, in part because it is easy to understand and in many cases compute.

In the context of random walks, we have the following elegant formula for the Minkowski dimension of the range of a transient random walk on $\mathbb{Z}^d$.

Theorem 6.1. *Let X denote a transient random walk on $\mathbb{Z}^d$, with Green's function g, as before. Then, with probability one,*

$$\overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) = \gamma_c,$$

where

$$\gamma_c := \inf\left\{\gamma \in (0\,,d) : \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \frac{g(0\,,x)}{\|x\|^\gamma} < \infty\right\}, \tag{6.1}$$

where $\inf\varnothing := d$.

The proof hinges on the analysis of the *0-potential measure*,

$$U(A) := \sum_{n=0}^\infty P^0\{X_n \in A\} = \sum_{x\in A} g(0\,,x), \tag{6.2}$$

defined for all $A \subset \mathbb{R}^d$. Because X is transient, the set function U is a Radon measure on $\mathbb{R}^d$. Since $g(x\,,y) = g(0\,,y-x)$ for all $x,y \in \mathbb{Z}^d$, it follows that for all $A \subset \mathbb{R}^d$,

$$E^0\left[\mathrm{card}\left(\mathcal{R}_X \cap A\right)\right] = \sum_{x\in A} P^0\left\{X_k = x \text{ for some } k \geqslant 0\right\} = \frac{U(A)}{g(0\,,0)}, \tag{6.3}$$

thanks to a combination of Tonelli's theorem and (5.1).

The following simple argument implies the first half of Theorem 6.1.

Proof of Theorem 6.1: Upper bound. We first prove that, with probability one,

$$\overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) \leqslant \gamma_c. \tag{6.4}$$

The more involved converse bound will be proved later.

Chebyshev's inequality and (6.3) together show that for all real numbers $\gamma > 0$ and integers $k \geqslant 1$,

$$P^0\left\{\mathrm{card}(\mathcal{R}_X \cap \mathcal{S}_k) \geqslant 2^{k\gamma}\right\} \leqslant \frac{2^{-k\gamma}U(\mathcal{S}_k)}{g(0\,,0)}. \tag{6.5}$$

Because $g(x\,,y) \leqslant g(0\,,0) < \infty$ for all $x,y \in \mathbb{Z}^d$—see (5.1)—there exists a finite constant b such that $U(\mathcal{S}_k) \leqslant b2^{kd}$ for all $k \geqslant 1$. Therefore, the sum over k of the right-hand side of (6.5) is always finite when $\gamma > d$. If $\gamma \in (0\,,d)$ is such that the right-hand side of (6.5) forms a summable series [indexed by k], then so does the left-hand side. The Borel–Cantelli lemma shows that for any such value of γ the random variable L_γ defined by

$$L_\gamma := \sup_{k\in\mathbb{N}}\left\{\frac{\mathrm{card}(\mathcal{R}_X \cap \mathcal{S}_k)}{2^{k\gamma}}\right\}.$$

is a.s. finite. In particular,

$$\mathrm{card}(\mathcal{R}_X \cap \mathcal{V}_k) \leqslant \mathrm{card}(\mathcal{V}_0) + L_\gamma \sum_{j=1}^k 2^{j\gamma} \leqslant 2^\gamma(L_\gamma \vee 4^d)2^{k\gamma}, \tag{6.6}$$

for all $k \geqslant 1$. This proves that

$$\limsup_{n\to\infty} n^{-1} \log_2 \left[\text{card}\left(\mathcal{R}_X \cap \mathcal{V}_n\right)\right] \leqslant \gamma \qquad \text{a.s.},$$

whence $\overline{\text{Dim}}_{\text{M}}(\mathcal{R}_X) \leqslant \gamma$ a.s. for such a γ. Optimize over all such γ's in order to see that

$$\overline{\text{Dim}}_{\text{M}}(\mathcal{R}_X) \leqslant \inf \left\{ \gamma \in (0\,,d) : \sum_{k=1}^{\infty} 2^{-k\gamma} U(\mathcal{S}_k) < \infty \right\}, \tag{6.7}$$

where $\inf \varnothing := d$. To finish, note that if $x \in \mathcal{S}_k$ then $\|x\| \geqslant 2^{k-1}$, whence

$$\begin{aligned}
\sum_{k=1}^{\infty} 2^{-k\gamma} U(\mathcal{S}_k) &= \sum_{k=1}^{\infty} 2^{-k\gamma} \sum_{x\in\mathcal{S}_k} g(0\,,x) \\
&\geqslant 2^{-\gamma} \sum_{k=1}^{\infty} \|x\|^{-\gamma} \sum_{x\in\mathcal{S}_k} g(0\,,x) \qquad (6.8)\\
&= 2^{-\gamma} \sum_{x\in\mathbb{Z}^d\setminus\mathcal{V}_0} \frac{g(0\,,x)}{\|x\|^{\gamma}}.
\end{aligned}$$

This and (6.7) together imply (6.4). □

For the remaining, more challenging, direction of Theorem 6.1 we need to know that the measure U is *volume-doubling*. That is the essence of the following result.

Proposition 6.2. $U(\mathcal{V}_{n+1}) \leqslant 4^d U(\mathcal{V}_n)$ *for all* $n \geqslant 0$.

This is a volume-doubling result because $\mathcal{V}_n = 2\mathcal{V}_{n-1}$. See Khoshnevisan and Xiao [10] for a corresponding result for Lévy processes on $\mathbb{R}^d$.

Proof. The proposition holds trivially when $n = 0$. Therefore, we will concentrate on the case $n \geqslant 1$ from now on.

We begin with a familiar series of random-walk computations. Choose and fix an integer $n \geqslant 1$ and some $x \in \mathbb{Z}^d$. Then, we apply the strong Markov property at $\tau := \inf\{k \geqslant 0 : X_k \in x + \mathcal{V}_{n-1}\}$ [$\inf \varnothing := +\infty$] in order to see that

$$U(x + \mathcal{V}_{n-1}) = E^0\left[U\left(-X_\tau + x + \mathcal{V}_{n-1}\right);\, \tau < \infty\right].$$

Since $-X_\tau + x \in -\mathcal{V}_{n-1}$ P^0-a.s. on $\{\tau < \infty\}$, and $-\mathcal{V}_{n-1} + \mathcal{V}_{n-1} = \mathcal{V}_n$, this readily yields the "shifted-ball inequality,"

$$\sup_{x\in\mathbb{Z}^d} U(x + \mathcal{V}_{n-1}) \leqslant U(\mathcal{V}_n) \qquad \text{for all } n \geqslant 1. \tag{6.9}$$

Eq. (6.9) becomes obvious if "$\sup_{x\in\mathbb{Z}^d}$" were replaced by "$\sup_{x\in\mathcal{V}_{n-1}}$." The strong Markov property of X was needed in order to establish this improvement.

Armed with (6.9) we proceed in a standard way: We can always find 4^d integer points $x_1, \ldots, x_{4^d} \in \mathbb{Z}^d$ such that

$$\mathcal{V}_{n+1} = \bigcup_{j=1}^{4^d} (x_j + \mathcal{V}_{n-1}),$$

for all $n \geqslant 1$, where the union is a disjoint one. Thus,

$$U(\mathcal{V}_{n+1}) = \sum_{j=1}^{4^d} U(x_j + \mathcal{V}_{n-1}) \leqslant 4^d \sup_{x\in\mathbb{Z}^d} U(x + \mathcal{V}_{n-1}).$$

The proposition follows from this and (6.9). □

Next we rewrite γ_c—see (6.1)—in a slightly different form. We will be ready to complete the proof of Theorem 6.1 once that task is done.

Proposition 6.3. $\gamma_c = \limsup_{n\to\infty} n^{-1}\log_2 U(\mathcal{V}_n)$.

Proof. If $x \in \mathcal{S}_k$, then $\|x\| \leqslant d^{1/2}2^k$. Therefore,

$$\sum_{k=1}^{\infty} 2^{-k\gamma} U(\mathcal{S}_k) \leqslant d^{\gamma/2} \sum_{x\in\mathbb{Z}^d\setminus\mathcal{V}_0} \frac{g(0\,,x)}{\|x\|^{\gamma}}.$$

Therefore, we can infer from (6.8) that

$$\gamma_c = \inf\left\{\gamma \in (0\,,d) :\ \sum_{k=1}^{\infty} 2^{-k\gamma} U(\mathcal{S}_k) < \infty\right\}. \tag{6.10}$$

We apply (6.10) to rewrite γ_c once more time: If $\gamma > \gamma_c$, then $U(\mathcal{S}_k) = o(2^{k\gamma})$ as $k\to\infty$. If on the other hand $\gamma \in (0\,,\gamma_c)$, then we can argue by contraposition to see that, for every fixed $\epsilon > 0$, $U(\mathcal{S}_k) > 2^{k(\gamma-\epsilon)}$ for infinitely-many integers k. This means that

$$\gamma_c = \limsup_{n\to\infty} n^{-1}\log_2 U(\mathcal{S}_n). \tag{6.11}$$

Now we prove the proposition.

The assertion of the proposition is that $\gamma_c = \gamma_c'$, where

$$\gamma_c' := \limsup_{n\to\infty} n^{-1}\log_2 U(\mathcal{V}_n).$$

On one hand, (6.11) implies that $\gamma_c \leqslant \gamma_c'$. If, on the other hand, $\vartheta > \gamma_c$ is an arbitrary finite number, then there exists a finite constant L_ϑ such that $U(\mathcal{S}_k) \leqslant L_\vartheta 2^{k\vartheta}$ for all integers $k \geqslant 1$. In particular,

$$U(\mathcal{V}_n) \leqslant \mathrm{card}(\mathcal{V}_0) + L_\vartheta \sum_{k=1}^{n} 2^{k\vartheta} \qquad \text{for all } n \geqslant 1,$$

whence follows that $U(\mathcal{V}_n) = O(2^{n\vartheta})$ as $n\to\infty$. Since this is true for all $\vartheta > \gamma_c$, it follows that $\gamma_c' \leqslant \gamma_c$, as was promised. □

Proof of Theorem 6.1: Lower bound. It remains to prove that

$$\overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) \geqslant \gamma_c \qquad \text{a.s.}, \tag{6.12}$$

where γ_c was defined in (6.1). If $\gamma_c = 0$, then we are done. Therefore, from now on we assume without loss of generality that

$$\gamma_c > 0. \tag{6.13}$$

Define

$$\tau(x) := \inf\{n \geqslant 0 :\ X_n = x\},$$

for all $x \in \mathbb{Z}^d$ [$\inf\varnothing := +\infty$]. Since $\mathrm{card}(\mathcal{R}_X \cap A) = \mathrm{card}\{x \in A :\ \tau(x) < \infty\}$, Tonelli's theorem implies that

$$\begin{aligned} &E^0\left(|\mathrm{card}\,(\mathcal{R}_X\cap\mathcal{V}_n)|^2\right)\\ &= E^0\left[\mathrm{card}\,(\mathcal{R}_X\cap\mathcal{V}_n)\right] + 2\mathop{\sum\sum}_{\substack{x,y\in\mathcal{V}_n\\ x\neq y}} P^0\left\{\tau(x)<\tau(y)<\infty\right\}\\ &= E^0\left[\mathrm{card}\,(\mathcal{R}_X\cap\mathcal{V}_n)\right] + 2\mathop{\sum\sum}_{\substack{x,y\in\mathcal{V}_n\\ x\neq y}} P^0\{\tau(x)<\infty\}P^x\{\tau(y)<\infty\}, \end{aligned} \tag{6.14}$$

thanks to the strong Markov property. If $x \in \mathcal{V}_n$ and $n \geqslant 1$ are held fixed, then

$$\sum_{y \in \mathcal{V}_n \setminus \{x\}} P^x\{\tau(y) < \infty\} = \frac{U(\mathcal{V}_n - x)}{g(0\,,0)} \leqslant \frac{U(\mathcal{V}_{n+1})}{g(0\,,0)},$$

since $\mathcal{V}_n - x \subset \mathcal{V}_n - \mathcal{V}_n = \mathcal{V}_{n+1}$. Therefore, (6.14) implies that

$$\begin{aligned} E^0\left(|\mathrm{card}\,(\mathcal{R}_X \cap \mathcal{V}_n)|^2\right) &\leqslant E^0\left[\mathrm{card}\,(\mathcal{R}_X \cap \mathcal{V}_n)\right] + \frac{2U(\mathcal{V}_n)U(\mathcal{V}_{n+1})}{[g(0\,,0)]^2} \\ &\leqslant E^0\left[\mathrm{card}\,(\mathcal{R}_X \cap \mathcal{V}_n)\right] + 2^{1+2d}\left\{E^0\left[\mathrm{card}\,(\mathcal{R}_X \cap \mathcal{V}_n)\right]\right\}^2, \end{aligned} \tag{6.15}$$

thanks to (6.3) and Proposition 6.2. Because

$$\mathrm{E}[\mathrm{card}(\mathcal{R}_X \cap \mathcal{V}_n)] = \frac{U(\mathcal{V}_n)}{g(0\,,0)},$$

Eq. (6.15) and the Paley–Zygmund inequality together yield the following: For infinitely many values of $n \in \mathbb{N}$,

$$\begin{aligned} P^0\left\{\mathrm{card}\,(\mathcal{R}_X \cap \mathcal{V}_n) > \frac{U(\mathcal{V}_n)}{2g(0\,,0)}\right\} &\geqslant \frac{\left\{E^0\left[\mathrm{card}\,(\mathcal{R}_X \cap \mathcal{V}_n)\right]\right\}^2}{4E^0\left(|\mathrm{card}\,(\mathcal{R}_X \cap \mathcal{V}_n)|^2\right)} \\ &\geqslant \frac{1}{4(1+2^{1+d})} \\ &:= \varrho(d). \end{aligned}$$

The last part holds since (6.13) implies that $\mathrm{E}[\mathrm{card}(\mathcal{R}_X \cap \mathcal{V}_n)] \geqslant 1$ for infinitely-many integers $n > 1$. The preceding displayed inequality and Proposition 6.3 together imply that $\overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) \geqslant \gamma_c$, with probability at least $\varrho(d) > 0$. Since $\overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) = \overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X \cap \mathcal{V}_N^c)$ for all $N \geqslant 1$, an application of the Hewitt–Savage 0–1 law shows the desired result that $\overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) \geqslant \gamma_c$ almost surely. □

7 Concluding remarks and open problems

Corollary 5.2 succeeds in yielding a formula for $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F)$ for every recurrent set $F \subset \mathbb{Z}^d$, though it is difficult to work with that formula. We do not expect a simple formula for $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F)$ when F is a general recurrent set in $\mathbb{Z}^d$. In fact, it is not even easy to decide whether or not a general set F is recurrent, as we have seen already. However, one can hope for simpler descriptions of $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X \cap F)$ when $F = \mathbb{Z}^d$. In this section we conclude with a series of remarks, problems, and conjectures that have these comments in mind.

Question (1.1) was in part motivated by its "local variation," which had been open since the mid-to-late 1960's [21], and possibly earlier. Namely, let $\{y(t)\}_{t\geqslant 0}$ be a Lévy process in $\mathbb{R}^d$. The local version of (1.1) asks, "what is the ordinary Hausdorff dimension $\dim_{\mathrm{H}}$ of the range $y(\mathbb{R}_+) := \cup_{t\geqslant 0}\{y(t)\}$?" This question was answered several years later by Khoshnevisan, Xiao, and Zhong [13, Corollary 1.8], who showed among other things that $\dim_{\mathrm{H}}(y(\mathbb{R}_+))$ is a.s. equal to an index that was introduced earlier in Pruitt [21] as part of the solution to the very same question. Under a quite mild regularity condition, it has been shown that the general formula for $\dim_{\mathrm{H}}(y(\mathbb{R}_+))$ reduces to the following [12, (1.19) of Theorem 1.5]:

$$\dim_{\mathrm{H}}(y(\mathbb{R}_+)) = \sup\left\{\gamma \in (0\,,d):\ \int_{\mathbb{R}^d} \frac{u(x)}{\|x\|^\gamma}\,\mathrm{d}x < \infty\right\} \quad \text{a.s.,} \tag{7.1}$$

where u denotes the 1-potential kernel of y. Khoshnevisan and Xiao [11, eq. (1.4)] find an alternative Fourier-analytic formula.

If we proceed purely by analogy, then we might guess from (6.1) and (7.1) the following formula for the macroscopic Hausdorff dimension of the range $\mathcal{R}_X$ of our random walk X on $\mathbb{Z}^d$:

$$\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) = \gamma_c \qquad \text{a.s.} \tag{7.2}$$

In principle, we ought to be able to decide whether or not (7.2) is correct, based solely on Corollary 5.2. But we do not know how to do that at this time mainly because it is quite difficult to compute $p_c(\mathbb{Z}^d\,;0)$ when X is a general transient random walk. Instead, we are able to only offer

Conjecture 7.1. $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) = \gamma_c$ *a.s. for every transient random walk on* $\mathbb{Z}^d$*, where* γ_c *was defined in* (6.1)*.*

Because of Theorem 6.1, Conjecture 7.1 is equivalent to the assertion that $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X) = \overline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X)$ a.s. It is known that the ordinary [microscopic] Hausdorff dimension of the range of a Lévy process is always equal to its ordinary [microscopic] *lower* Minkowski dimension, and not always the upper Minkowski dimension. If Conjecture 7.1 were correct, then it would suggest that large-scale dimension theory of random walks is somewhat different from its small-scale counterpart. Our next Problem is an attempt to understand this difference better.

Barlow and Taylor [1, 2] have introduced two other notions of macroscopic dimension that are related to our present interests. Namely, they define the [macroscopic] *lower Minkowski dimension* of $A \subset \mathbb{Z}^d$ and the *lower Hausdorff dimension* of $A \subset \mathbb{Z}^d$ respectively as[5]

$$\underline{\mathrm{Dim}}_{\mathrm{M}}(A) := \liminf_{n\to\infty} n^{-1} \log \mathrm{card}(A \cap \mathcal{V}_n),$$
$$\underline{\mathrm{Dim}}_{\mathrm{H}}(A) := \inf\left\{\alpha > 0 : \lim_{k\to\infty} \mathcal{N}_\alpha(A\,,\mathcal{S}_k) = 0\right\}.$$

One has $\underline{\mathrm{Dim}}_{\mathrm{H}}(A) \leqslant \mathrm{Dim}_{\mathrm{H}}(A)$ and $\underline{\mathrm{Dim}}_{\mathrm{M}}(A) \leqslant \overline{\mathrm{Dim}}_{\mathrm{M}}(A)$ for all $A \subseteq \mathbb{Z}^d$.

It is easy to obtain a nontrivial upper bound for the lower Minkowski dimension of $\mathcal{R}_X$, valid for every transient random walk X on $\mathbb{Z}^d$. Namely, by (6.3) and Fatou's lemma,

$$\mathrm{E}\left[\liminf_{n\to\infty} 2^{-n\gamma}\,\mathrm{card}(\mathcal{R}_X \cap \mathcal{V}_n)\right] \leqslant \liminf_{n\to\infty} 2^{-n\gamma} U(\mathcal{V}_n),$$

for every $\gamma \in [0\,,\infty)$. From this we readily can deduce that

$$\underline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) \leqslant \liminf_{n\to\infty} n^{-1} \log U(\mathcal{V}_n) \qquad \text{a.s.} \tag{7.3}$$

We believe that this is a sharp bound, and thus propose the following.

Conjecture 7.2. *With probability one,*

$$\underline{\mathrm{Dim}}_{\mathrm{H}}(\mathcal{R}_X) = \underline{\mathrm{Dim}}_{\mathrm{M}}(\mathcal{R}_X) = \liminf_{n\to\infty} n^{-1} \log U(\mathcal{V}_n).$$

Admittedly, we have not tried very hard to prove this, but it seems to be a natural statement. There are two other good reasons for our interest in Conjecture 7.2. First of all, it suggests that, as far as random walks and their analogous Lévy processes are concerned, the more natural choice of "macroscopic Hausdorff dimension" is $\underline{\mathrm{Dim}}_{\mathrm{H}}$ and not $\mathrm{Dim}_{\mathrm{H}}$, in contrast with the proposition of [1, 2]. Also, if Conjecture 7.2 were true, then together with Theorem 6.1 and Proposition 6.3 it would imply that regardless of

[5] Barlow and Taylor write $\dim_{\mathrm{LM}}$ and $\dim_{\mathrm{L}}$ in place of our $\underline{\mathrm{Dim}}_{\mathrm{M}}$ and $\underline{\mathrm{Dim}}_{\mathrm{H}}$.

whether or not Conjecture 7.1 is true, $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X)$ always lies in the non-random interval $[\liminf_{n\to\infty} n^{-1}\log U(\mathcal{V}_n)\,,\limsup_{n\to\infty} n^{-1}\log U(\mathcal{V}_n)]$. The extrema of this interval are typically not hard to compute; therefore, we at least will have easy-to-compute bounds for $\mathrm{Dim}_{\mathrm{H}}(\mathcal{R}_X)$.

Let us state a third conjecture that is motivated also by Conjecture 7.1.

Choose and fix an arbitrary integer $N \geqslant 1$, and define $X^{(1)},\ldots,X^{(N)}$ to be N independent copies of a symmetric, transient random walk X on $\mathbb{Z}^d$ whose Green's function satisfies the Barlow–Taylor condition (1.3) for some $\alpha \in (0\,,2]$. We can define an N-parameter *additive random walk* $\mathfrak{X} := \{\mathfrak{X}(\boldsymbol{n})\}_{\boldsymbol{n}\in\mathbb{Z}_+^N}$ as follows [9, Ch. 4]: For every $\boldsymbol{n} := (n_1\,,\ldots,n_N) \in \mathbb{Z}_+^N$,

$$\mathfrak{X}(\boldsymbol{n}) := X^{(1)}_{n_1} + \cdots + X^{(N)}_{n_N}.$$

Let $\mathcal{R}_{\mathfrak{X}} := \cup_{\boldsymbol{n}\in\mathbb{Z}_+^N}\{\mathfrak{X}(\boldsymbol{n})\}$ denote the *range* of the random field $\mathfrak{X}$.

Conjecture 7.3. *Suppose $d > \alpha N$ and $N > 1$. Then for all non-random $A \subset \mathbb{Z}^d$:*

1. *If $\mathrm{Dim}_{\mathrm{H}}(A) > d - \alpha N$, then $\mathcal{R}_{\mathfrak{X}} \cap A$ is a.s. unbounded; and*
2. *If $\mathrm{Dim}_{\mathrm{H}}(A) < d - \alpha N$, then $\mathcal{R}_{\mathfrak{X}} \cap A$ is a.s. bounded.*

Proposition 1.1 implies that Conjecture 7.3 is correct if N were replaced by 1; the case $N > 1$ has eluded our many attempts at solving this problem.

It is possible to adapt the arguments of [13] in order to derive Conjecture 7.1 from Conjecture 7.3. We skip the details of that argument. Instead, we conclude with two problems about the "continuous version" of Corollary 5.2, which we recall, contained our Hausdorff dimension formula for the range of a walk.

Problem 7.4. Let $\{y(t)\}_{t\geqslant 0}$ be a transient, but otherwise general, Lévy process on $\mathbb{R}^d$ whose characteristic exponent is Ψ, normalized as $\exp\{iz\cdot y(t)\} = \exp\{-t\Psi(z)\}$ for all $z \in \mathbb{R}^d$ and $t \geqslant 0$, to be concrete. Is there a formula for the a.s.-constant quantity $\mathrm{Dim}_{\mathrm{H}}(y(\mathbb{R}_+))$ that is solely in terms of Ψ?

Before we state our last question let us define the *upper Minkowski dimension* of a set $A \subseteq \mathbb{R}^d$ as follows: Define A' to be the union of all dyadic cubes $Q \in \mathsf{D}_0$ of sidelength one that intersect A.

Definition 7.5. *The* macroscopic upper Minkowski dimension $\overline{\mathrm{Dim}}_{\mathrm{M}}(A)$ *is defined, via the Barlow–Taylor upper Minkowski dimension, as $\overline{\mathrm{Dim}}_{\mathrm{M}}(A) := \overline{\mathrm{Dim}}_{\mathrm{M}}(A')$ for all $A \subset \mathbb{R}^d$.*

The same proof that worked for $A \subseteq \mathbb{Z}^d$ continues to work in order to show that $\mathrm{Dim}_{\mathrm{H}}(A) \leqslant \overline{\mathrm{Dim}}_{\mathrm{M}}(A)$ for all $A \subset \mathbb{R}^d$.

Although we have not checked all of the details, we believe that the method of proof of Theorem 6.1 can be adapted to the continuous setting in order to produce

$$\overline{\mathrm{Dim}}_{\mathrm{M}}(y(\mathbb{R}_+)) = \limsup_{n\to\infty} n^{-1}\log \mathbb{U}(\mathcal{V}_n) \qquad \text{a.s.},$$

where $\mathbb{U}(A) := \int_0^\infty \mathrm{P}\{y(s) \in A\}\,\mathrm{d}s$ for all Borel sets $A \subset \mathbb{R}^d$.

Problem 7.6. Is there an expression for $\overline{\mathrm{Dim}}_{\mathrm{M}}(y(\mathbb{R}_+))$ solely in terms of Ψ?

Conjecture 7.3 is likely to have a Lévy process version wherein the role of the $X^{(i)}$s are replaced by that of isotropic α-stable Lévy processes. We leave the statement [and perhaps also a proof!] to the interested reader.

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